Solution of Allen Hatcher

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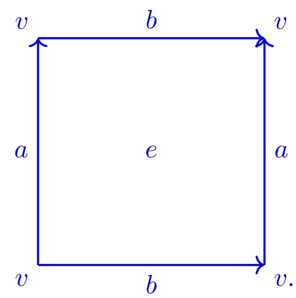
Contents

Chapter 0

Some Underlying Geometric Notions

Exercise 0.0.1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution. We know that the torus can be identified by the quotient space of



Suppose the fundamental domain is $[0,1] \times [0,1]$ □

Exercise 0.0.2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Solution. Define a map

$$\begin{aligned} \pi : \mathbb{R}^n - \{0\} &\to S^{n-1} \\ (x_1, \cdots, x_n) &\mapsto \frac{1}{\sqrt{x_1^2 + \cdots + x_n^2}}(x_1, \cdots, x_n). \end{aligned}$$

It is continuous because every coordinate component is continuous. Also, $\pi|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, therefore the map is a deformation retraction. □

Exercise 0.0.3. (i) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

(ii) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(iii) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution. (i) □

Exercise 0.0.4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{id}_X$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Solution. □

Exercise 0.0.5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subseteq U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

Solution. □

Exercise 0.0.6 (Exercise 0.10)**.** Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

Solution. (i) Suppose X is contractible, then there is a point x_0, and maps $h : X \to \{x_0\}$, $g : \{x_0\} \to X$ s.t. $g \circ h \simeq \mathrm{id}|_X$ and $h \circ g \simeq \mathrm{id}|_{\{x_0\}}$. We denote the homotopy as $F : X \times I \to X$ where $F|_{X\times\{0\}} = \mathrm{id}$ and $F|_{X\times\{1\}} = g \circ h$. For any $f : X \to Y$ where Y is an arbitrary space, let $y_0 = f(g(x_0))$, and let $G := f \circ F$. Thus $G : X \times I \to Y$ is continuous since it is the composition of two continuous maps. $G|_{X\times\{0\}} = f \circ \mathrm{id} = f$ and $G|_{X\times\{1\}} = f \circ g \circ h$. But $f \circ g \circ h(X) = y_0$. Therefore $f : X \to Y$ is nullhomotopic.

Conversely, put $Y = X$, then we know that $\mathrm{id} : X \to X$ is nullhomotopic. That is, we have a constant map $g : X \to X$ and a homotopy $F : X \times I \to X$ s.t. $F|_{X\times\{0\}} = \mathrm{id}$ and $F|_{X\times\{1\}} = g$. g being a constant map means $g(X) = \{x_0\}$ for some $x_0 \in X$, so we say g is a map $X \to \{x_0\}$ and define $f : \{x_0\} \to X$, $x_0 \mapsto x_0$. Thus $g \circ f = \mathrm{id}_{\{x_0\}}$ and $f \circ g = g$. The existence of F implies $f \circ g \simeq \mathrm{id}$.

(ii) Suppose X is contractible, then there is a point x_0, and maps $h : X \to \{x_0\}$, $g : \{x_0\} \to X$ s.t. $g \circ h \simeq \mathrm{id}|_X$ and $h \circ g \simeq \mathrm{id}|_{\{x_0\}}$. We denote the homotopy as $F : X \times I \to X$ where $F|_{X\times\{0\}} = \mathrm{id}$ and $F|_{X\times\{1\}} = g \circ h$. Define $G : Y \times I \to X$, $(y, t) \mapsto F(f(y), t)$. Hence $G|_{Y\times\{0\}} = F(f(y), 0) = f(y)$ and $G|_{X\times\{1\}} = F(f(y), 1) = h(g(f(y))) = h(x_0)$. Thus, $f : X \to Y$ is nullhomotopic.

Conversely, put $Y = X$, then we know that $\mathrm{id} : X \to X$ is nullhomotopic. That is, we have a constant map $g : X \to X$ and a homotopy $F : X \times I \to X$ s.t. $F|_{X\times\{0\}} = \mathrm{id}$ and $F|_{X\times\{1\}} = g$. g being a constant map means $g(X) = \{x_0\}$ for some $x_0 \in X$, so we say g is a map $X \to \{x_0\}$ and define $f : \{x_0\} \to X$, $x_0 \mapsto x_0$. Thus $g \circ f = \mathrm{id}_{\{x_0\}}$ and $f \circ g = g$. The existence of F implies $f \circ g \simeq \mathrm{id}$. □

Exercise 0.0.7. Given positive integers v, e, and f satisfying $v - e + f = 2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells.

Solution. We do induction on v. Notice that v is at least 1, and f is at least 1 because S^2 is of dimension 2.

For $v = 1$, □

Exercise 0.0.8 (Exercise 0.20)**.** Show that the subspace $X \subseteq \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Exercise 0.0.9 (Exercise 0.23)**.** Show that a CW complex X is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Solution. Suppose $X = A \cup B$ and suppose $A \cap B$ is contractible. Hence by the first homotopy equivalence criterion, $\{*\} \simeq B \simeq B/A \cap B$. The map $\bar{\varphi} : X \to B/A \cap B$ induces a natural map

$$\varphi : X/A \to B/A \cap B,$$

where $\bar{\varphi}$ maps every point $x \in X - A$ to x itself in $B/A \cap B$, and sends A to $A \cap B/A \cap B$, i.e. we have the following

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\bar{\varphi}"] & \\ X/A \arrow[r, "\varphi"] & B/A \cap B. \end{tikzcd}$$

By the definition of quotient topology, φ is continuous. In fact, if U is an open set in $B/A \cap B$, then $\varphi^{-1}(U) = \pi_1 \circ \bar{\varphi}^{-1} \circ \pi_2^{-1}(U)$ is also open. Similarly, the map $\bar{\psi} : B \to X/A$ induces a natural map

$$\psi : B/A \cap B \to X/A,$$

where $\bar{\psi}$ maps every point $x \in B - A$ to x itself in X/A, and sends $A \cap B$ to A/A. Also we have

$$\begin{tikzcd} B \arrow[d, "\pi"'] \arrow[dr, "\bar{\psi}"] & \\ B/A \cap B \arrow[r, "\psi"] & X/A. \end{tikzcd}$$

The same argument shows ψ is continuous. Since $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$, we have a homeomorphism

$$X/A \cong B/A \cap B.$$

Again by the first homotopy equivalence criterion, $X \simeq X/A$ since A is contractible. Hence $X \simeq X/A \cong B/A \cap B \simeq A \simeq \{*\}$. □

Exercise 0.0.10 (Exercise 0.28)**.** Show that if (X_1, A) satisfies the homotopy extension property, then so does every pair $(X_0 \sqcup_f X_1, X_0)$ obtained by attaching X_1 to a space X_0 via map $f : A \to X_0$.

Remark. This is a more general result: cofibrations are stable under taking pushouts.

Solution. Since (X_1, A) satisfies the homotopy extension property, we have a retraction $r : X_1 \times I \to X_1 \times \{0\} \cup A \times I$. By the quotient map, we have

$$\begin{tikzcd}
(X_0 \sqcup X_1) \times I \arrow[r, "\mathrm{id}_{X_0} \times r"] \arrow[d, "\bar{f}"] & (X_0 \times I) \sqcup (X_1 \times \{0\} \cup A \times I) \arrow[d, "\tilde{f}"] \\
(X_0 \sqcup_f X_1) \times I \arrow[r, dashed] & (X_0 \times I) \sqcup_f (X_1 \times \{0\})
\end{tikzcd}$$

where $\bar{f}$ attaches X_1 to X_0 and leaves I stable, and similarly $\tilde{f}$ attaches $X_0 \times \{0\}$ and $A \times I$ to $X_0 \times I$. We say that guarantees a map $R : (X_0 \sqcup_f X_1) \times I \to (X_0 \times I) \sqcup_f (X_1 \times \{0\})$ which makes the diagram commute. R can be defined as following: for any $(x, t) \in (X_0 \sqcup_f X_1) \times I$, there is a $(\bar{x}, t)$ s.t. $\bar{f}(\bar{x}, t) = (x, t)$. Hence define $R(x, t) = \tilde{f} \circ (\mathrm{id}_{X_0} \times r)(\bar{x}, t)$.

The map is well-defined. Actually, if $(x_0, t), (x_1, t)$ are two different points that $\bar{f}(x_i, t) = (x, t)$, then $x \in f(A) \subseteq X_0$. W.l.o.g., assume $x_i \in X_i$. It suffices to prove $\tilde{f} \circ (\mathrm{id}_{X_0} \times r)(x_0, t) = \tilde{f} \circ (\mathrm{id}_{X_0} \times r)(x_1, t)$. But $\tilde{f} \circ (\mathrm{id}_{X_0} \times r)$ and $\bar{f}$ leave $X_0 \times I$ stable, so $x_0 = x$ and $\tilde{f} \circ (\mathrm{id}_{X_0} \times r)(x_0, t) = (x_0, t) = (x, t)$. On the other hand, $\tilde{f} \circ (\mathrm{id}_{X_0} \times r)(x_1, t) = (x_1, t) \in A \times I$. Thus $f(x_1) = x$ implies $\tilde{f}(x_1, t) = (x, t)$.

Finally, R is a retract. It is continuous since the definition of quotient topology. We have seen it remains stable on $X_0 \times I$. For any $(x_1, 0) \in (X_1 - f(A)) \times \{0\} \subseteq (X_0 \sqcup_f X_1) \times I$, we still have only one $(\bar{x_1}, 0) \in (X_1 - A) \times I$ s.t. $\bar{f}(\bar{x_1}, t) = (x_1, t)$. Actually it is itself. But $r|_{X_1 \times \{0\}} = \mathrm{id}$, hence $\tilde{f} \circ (\mathrm{id}_{X_0} \times r)(\bar{x_1}, 0) = (x_1, 0) \in X_1 \times \{0\}$.

In conclusion, $(X_0 \times I) \sqcup_f (X_1 \times \{0\})$ is a retract of $(X_0 \sqcup_f X_1) \times I$ so $(X_0 \sqcup_f X_1, X_0)$ satisfies the homotopy extension property. □

Chapter 1

The Fundamental Group

1.1 Basic Constructions

Exercise 1.1.1 (Exercise 1.1.7)**.** Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]

Solution. Let

$$\begin{aligned} F :& S^1 \times I \times I \to S^1 \times I \\ & (\theta, s, t) \mapsto (\theta + 2\pi st, s). \end{aligned}$$

Hence $F|_{S^1\times I\times\{0\}} = \mathrm{id}$ and $F|_{S^1\times I\times\{1\}} = f$, and F is continuous since every component of F is continuous, therefore F is the homotopy from id to f. Again by definition, $F|_{S^1\times\{0\}\times I} = \mathrm{id}$ so F is stationary on one of the boundary circle.

Suppose we have a homotopy $G : S^1 \times I \times I \to S^1 \times I$ giving $f_t \simeq \mathrm{id}$, s.t. G is stationary on both of the boundary circles. Define a family of paths $\gamma_{\theta_0} : I \to S^1 \times I$ by $s \mapsto (\theta_0, s)$. Explicitly the conditions are $G(\gamma_\theta(s), 0) = \gamma_\theta(s)$, $G(\gamma_\theta(s), 1) = f \circ \gamma_\theta(s)$, and $G(\gamma_\theta(0), t) = (\theta, 0)$, $G(\gamma_\theta(1), t) = (\theta, 1)$. Hence we have a homotopy from path γ_{θ_0} to $f \circ \gamma_{\theta_0}$. Then we consider the projection $\pi : S^1 \times I \to S^1 \times \{0\}$, $(\theta, s) \mapsto (\theta, 0)$. Then $\pi \circ \gamma_\theta$ and $\pi \circ f \circ \gamma_\theta$ are homotopy equivalent loops of S^1. However, $\pi \circ \gamma_\theta$ is a point so $[\pi \circ \gamma_\theta] = 0$ but $[\pi \circ f \circ \gamma_\theta] = 1$ since the projection is surjective, which leads a contradiction. □

Exercise 1.1.2 (Exercise 1.1.12)**.** Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Solution. Since $\pi_1(S^1) \cong \mathbb{Z}$, every homomorphism $\varphi_* : \pi_1(S^1) \to \pi_1(S^1)$ is uniquely determined by $\varphi_*(1)$ since $\varphi_*(n) = n\varphi_*(1)$. Let $k = \varphi_*(1)$, then construct

$$\begin{aligned} \varphi :& S^1 \to S^1 \\ & z = (\cos\theta, \sin\theta) \mapsto e^{2\pi ikz} = (\cos k\theta, \sin k\theta) \end{aligned}$$

then is suffices to prove φ_* is the homomorphism induced by φ. Denote the generator of $\pi_1(S^1)$ as $[\omega(t)]$ where $\omega : I \to S^1, t \mapsto (\cos t, \sin t)$. Hence

$$\varphi_*([\omega]) = [\varphi \circ \omega] = [(\cos k\theta, \sin k\theta)] = [\omega]^k.$$

Therefore φ_* is the induced homomorphism. □

Exercise 1.1.3 (Exercise 1.1.16)**.** Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1 , show that $f_*\beta_h = \beta_{fh} f_*$ in the diagram

$$\begin{tikzcd}
\pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"] & \pi_1(X, x_0) \arrow[d, "f_*"] \\
\pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"] & \pi_1(Y, f(x_0)).
\end{tikzcd}$$

Solution. Suppose $[\omega]$ is an element in $\pi_1(X, x_1)$, then

$$\beta_{fh} \circ f_*([\omega]) = \beta_{fh}([f \circ \omega]) = [(fh) \circ f \circ \omega \circ \bar{fh}] = f_*[h \circ \omega \circ \bar{h}] = f_*\beta_h([\omega]).$$

Hence the diagram commutes. □

Exercise 1.1.4 (Exercise 1.1.16)**.** Show that there are no retractions $r : X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ with A any subspace homoeomorphic to S^1.
2. $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
3. $X = S^1 \times D^2$ and A the circle shown in the book.
4. $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
5. X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
6. X the Möbius band and A its boundary circle.

Solution. Suppose we have a retraction $r : X \to A$, then Proposition 1.17 tells us that the inclusion $i : A \hookrightarrow X$ induces an injection $i_* : \pi_1(A, x_0) \hookrightarrow \pi_1(X, x_0)$.

(i) Since X is contractible ($F : X \times I \to X, (x, t) \mapsto tx$), $\pi_1(X, x_0)$ is trivial. But $\pi_1(A, x_0)$ is $\mathbb{Z}$ so we cannot have an injection.

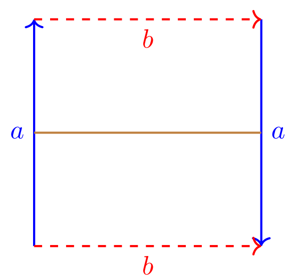
(ii) D^2 is contractible, so $\pi_1(X, x_0) = \pi_1(S^1 \times D^2, x_0) = \pi_1(S^1, x_0) \times \pi_1(D^2, x_0) = \mathbb{Z}$. But $\pi_1(A, x_0) = \pi_1(S^1 \times S^1, x_0) = \mathbb{Z} \times \mathbb{Z}$, so there is no injection.

(iii) We know that $\pi_1(A, x_0) \cong \mathbb{Z}$. Suppose $\omega : I \to A$ is a parametrization where $\omega(0) = \omega(1)$, so $[\omega]$ is one of the generators. But ω is contractible in X since $A = \partial D^1$ for some $D^1 \subseteq X$. So $[\omega]$ is 0 in $\pi_1(X, x_0)$. Hence the map i_* is never injective.

(iv) Suppose the base point is the intersection point of $X = D^2 \vee D^2$. Since each D^2 is contractible, $\pi_1(X, x_0) = \pi_1(D^2 \vee D^2, x_0) = 0$, but $\pi_1(A, x_0) = \pi_1(S^1 \vee S^1, x_0)(= \mathbb{Z} * \mathbb{Z})$. Instead of using van Kampen's theorem to prove it, it suffices to find an element which is not contractible. Suppose $[\omega]$ is an element in $\pi_1(S^1 \vee S^1, x_0)$ where $\omega : I \to S^1, t \mapsto e^{2\pi i t}$. Since $\pi_1(S^1, x_0) = \mathbb{Z}$, $[\omega]$ cannot be 0. Hence there cannot be an injection.

(v) First X is homomorphic to a circle through the point on the boundary identified. We can do this by shrink the surface from the two sides of the boundary to the circle we find. Thus $\pi_1(X, x_0) = \pi_1(S^1, x_0) = \mathbb{Z}$. Suppose α, β are two loops s.t. $[\alpha], [\beta]$ are two generators of the two different circles in $S^1 \vee S^1$. The inclusion gives that $[\alpha], [\beta]$ are homotopic since they are homotopically equivalent to the circle we find in X. But $[\alpha] \neq [\beta]$, so it is not an injection.

(vi) It is clear that $\pi_1(A, x_0)$ is $\mathbb{Z}$. Consider the retraction projecting the vertical lines onto the horizontal brown line (see the figure below),



this gives a homotopy $h : X \simeq S^1$ and thus $\pi_1(X) \cong \mathbb{Z}$ where a generator is the circle. In the figure above, the loop b is the subspace A, and suppose $\omega : I \to A$ is the loop b so that $[\omega]$ is a generator of $\pi_1(A)$. However $h \circ \omega$ gives the loop winding twice on S^1, so via the homotopy along the stripe the image of the generator of $\pi_1(A)$ is $[\omega]^2$, and hence $i_* : \pi_1(A) \to \pi_1(X)$ maps 1 to 2.

Suppose one has a retraction $r : X \to A$, then the composition

$$r_* \circ i_* : \pi_1(A) \to \pi_1(A)$$

is identity by the functoriality of π_1. However we've computed that i_* is not surjective so there is not a retraction. □

Exercise 1.1.5 (Exercise 1.1.18)**.** Using the technique in the proof of Proposition 1.14, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:

1. The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$;
2. For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.

Solution. Suppose $\omega : I \to X$ is a loop with $\omega(0) = \omega(1) = x_0 \in A$. Since e^n is open in X, hence $\omega^{-1}(e^n)$ is open in I. Assume $\omega^{-1}(e^n) = \bigcup_{i=1}^{\infty}(c_i, d_i)$, where $0 < c_i \leq d_i < 1$. (When $c_i = d_i$ we mean (c_i, d_i) is empty.) Thus $f(c_i), f(d_i) \in A$. Since A is path-connected, we have a path $f_i : [c_i, d_i] \to A$. Denote $g_i = \omega|_{[c_i,d_i]}$, then $F_i : f_i \simeq g_i$ since $\pi_1(D^n, x_0)$ is trivial. Thus, combine all these homotopies, we have a homotopy F s.t. $F|_{(c_i)} = F_i$ and $F = \mathrm{id}$ otherwise. Hence F is a homotopy from ω to a path in A.

(i) $S^1 \vee S^2 = S^1 \cup_f e^2$ where $f : \partial D^2 \to \{x_0\} \subseteq S^1$. Hence the inclusion $A \hookrightarrow S^1 \vee S^2$ induces a surjection $\mathbb{Z} \to \pi_1(S^1 \vee S^2)$. But $\mathbb{Z} \to \pi_1(S^1 \vee S^2)$ cannot be trivial since the circle S^1 is not nullhomotopic. Hence $\pi_1(S^1 \vee S^2) = \mathbb{Z}$.

(ii) Suppose ω is a path in X. Then $\omega(I)$ is compact in X and hence by proposition A.1 we know it is included in a finite CW complex, say X^n. Therefore by previous proof ω is homotopic to a path in X^1. Hence we have a surjective map $\pi_1(A, x_0) \to \pi_1(X, x_0)$. □

1.2 Van Kampen's Theorem

Exercise 1.2.1 (Exercise 1.2.4)**.** Let $X \subseteq \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

Solution. Consider the standard sphere $S^2 \subseteq \mathbb{R}^3$, then there is a radial deformation retraction $\mathbb{R}^3 - X \simeq S^2 \cap (\mathbb{R}^3 - X)$. Hence $\pi_1(\mathbb{R}^3 - X) = \pi_1(S^2 \cap (\mathbb{R}^3 - X))$ where $S^2 \cap (\mathbb{R}^3 - X)$ is the space punctured sphere with n holes. The punctured sphere with 1 hole is homeomorphic to Int D^2, hence $S^2 \cap (\mathbb{R}^3 - X)$ is homeomorphic to punctured disk with $2n - 1$ holes. Thus

$$\pi_1(S^2 \cap (\mathbb{R}^3 - X)) \cong \mathbb{F}_{n-1}$$

the free group generated by $2n - 1$ elements. □

Exercise 1.2.2 (Exercise 1.2.7)**.** Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Solution 1. Consider the CW structure on X given by. □

Solution 2. It is easy to see X is homotopic to the space Y consisting of a sphere and a diameter connecting the north pole and the south pole, since we have a deformation retract on the diameter into a point. Y is homotopic to $S^1 \vee S^2$ by Example 1.23 in Hatcher. Hence

$$\pi_1(X) \cong \pi_1(Y) \cong \pi_1(S^1 \vee S^2) = \mathbb{Z}$$

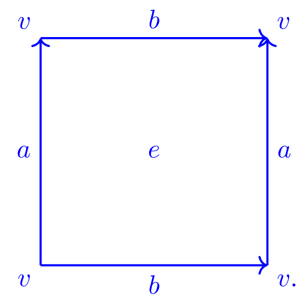
by van Kampen's theroem. □

Exercise 1.2.3 (Exercise 1.2.9)**.** In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C, and hence M_g does not retract onto C. But show that M_g does retract onto the nonseparating circle C' in the figure.

Solution. First we prove that M'_h does not retract onto its boundary circle C. By the argument in Chapter 0, the CW complex of M_h consists of 1 1-cells, $2g$ 1-cells and a 2-cell. M'_h is homeomorphic to cutting a hole inside the 2-cell. Hence M'_h is homotopic to $S^1 \vee \cdots \vee S^1$ of $2g$ copies. Thus $M'_h = \langle a_1 \rangle * \cdots * \langle a_{2g} \rangle$.

Then we suppose we have an retraction $r : M'_h \to C$, then r induces an injection $i_* : \pi_1(C) \to \pi_1(M'_h)$, where i is the inclusion $C \hookrightarrow M'_h$. Thus we have an injection of abelianization $(i_*)' : \pi_1(C)/\pi_1(C)' \to \pi_1(M'_h)/\pi_1(M'_h)'$. But $\pi_1(C) = \mathbb{Z}$ hence its abelianization is itself. But the loop C maps to $a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}$, whose image in $\pi_1(M'_h)$ happens to be a commutator. Hence $(i_*)'$ cannot be an injection, a contradiction.

To see M_g retracts on to C', we use the planar presentation of M_g, generated by $4g$ 1-cells (labelled by $a_1, b_1, \cdots, a_g, b_g$) and 1 2-cell. Take the quotient of $a_i \to a_1, b_i \to b_1$, we get a map $M_g \to M_1$. For M_1, the map mapping all horizontal line in the figure



into a point gives the retract to the circle represented by a, and the composition is the desired retraction. □

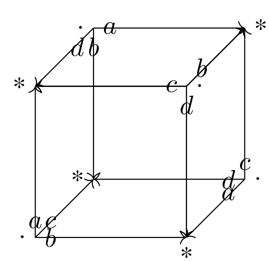
Exercise 1.2.4 (Exercise 1.2.10)**.** Consider two arcs α and β embedded in $D^2 \times I$ as shown in the picture. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.

Solution. First we notice that the complement is homeomorphic to I^3 minus two straight lines which do not intersect, since we can change the cylinder making the arcs straight. This can be deformation retracted to $I \times I$ minus two points, which is homotopic to $S^1 \vee S^1$. Hence the fundamental group $\pi_1(X)$ is $\mathbb{Z} * \mathbb{Z} = \langle a \rangle * \langle b \rangle$.

Then we consider how did the loop γ change. In the original figure, the loop γ splits the intersections of the straight lines in I^3, after the homeomorphism and homotopy, $[\gamma] = abab$ in $\pi_1(X)$, hence it is not homotopic to 0. □

Exercise 1.2.5 (Exercise 1.2.14)**.** Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaterion group of order 8.

Solution. I^3 can be naturally seen a CW complex with 8 1-cells, 6 2-cells and 1 3-cell. We first consider the 1-skeleton of X



Thus the 1-skeleton is a graph consisting two points $*, \cdot$ and four edges a, b, c, d from $\cdot$ to $*$ respectively. The quotient space has 1-skeleton generated by $ab^{-1}, ac^{-1}, ad^{-1}$, and the 2-cells are $ac^{-1}db^{-1}, ad^{-1}bc^{-1}, bc^{-1}ad^{-1}$. Denote $i = ab^{-1}, j = ac^{-1}, k = ad^{-1}$, then we have a group presentation of $\pi_1(X)$

$$\langle i, j, k \mid jki^{-1}, kij^{-1}, ijk^{-1} \rangle$$

by Prop 1.26. This is exactly a presentation of quaterion group of order 8. □

Exercise 1.2.6 (Exercise 1.2.15)**.** Given a space X with basepoint $x_0 \in X$, we may construct a CW complex $L(X)$ having a single 0-cell, a 1-cell e^1_γ for each loop γ in X based at x_0, and a 2-cell for each map τ of a standard triangle PQR into X taking the three vertices P, Q, R of the triangle to x_0. The 2-cell e^2_τ is attachd to the three 1-cells that are the loops obtained by reconstructing τ to the three oriented edges PQ, PR and QR. Show that the natural map $L(X) \to X$ induces an isomorphism $\pi_1(L(X)) \simeq \pi(X, x_0)$.

Solution. Since for every loop γ in X based at x_0, we have a 1-cell e^1_γ, the map $L(X) \to X$ induces a surjective map $\pi_1(L(X)) \twoheadrightarrow \pi(X, x_0)$. Then it suffices to prove that the kernel is 0. Suppose ω is a nullhomotopic loop in X, then we can find an open subset U of X s.t. $\omega([0,1]) \cup U \cong \omega([0,1]) \cup_f e^2$ where f maps the boundary of the disk to the image of ω by f. $\omega([0,1]) \cup U$ is homeomorphic to a triangle denoted by τ, hence we have an element $[PQ \circ QR \circ RP]$ in $\pi_1(L(X))$ s.t. $[\omega]$ is the image of $[PQ \circ QR \circ RP]$. But we have a 2-cell e^2_τ attached on it so $[PQ \circ QR \circ RP] = 0$. Therefore the induced homomorphism is an isomorphism. □

Exercise 1.2.7 (Exercise 1.2.16)**.** Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.

Solution. Let Y be a torus with two disjoint open disks removed. Since Y has a deformation retract on a circle with a diameter, we know $\pi_1(Y) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, and each boundary circle of the removed disk is a generator in $\mathbb{Z} \hookrightarrow \mathbb{Z} * \mathbb{Z}$. The surface of infinite genus consists of infinitely many copies of Y, attached on the boundary circles. By van Kampen's theorem, after gluing two copies of Y together, the fundamental group becomes $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Keep doing the gluing, we know the fundamental group is free on an infinite number of generators. □

Exercise 1.2.8 (Exercise 1.2.22)**.** In this exercise we describe an algorithm for computing a presentation of the fundamental group of the complement of a smooth or piecewise linear knot K in $\mathbb{R}^3$, called the *Wirtinger presentation*. To begin, we position the knot to lie almost flat on a table, so that K consists of finitely many disjoint arcs α_i where it intersects the table top together with finitely many disjoint arcs β_l where K crosses over itself. The configuration at such a crossing is shown in the first figure below. We build a 2 dimensional complex X that is a deformation retract of $\mathbb{R}^3 - K$ by the following three steps. First, start with the rectangle T formed by the table top. Next, just above each arc α_i place a long, thin rectangular strip R_i, curved to run parallel to α_i along the full length of α_i and arched so that the two long edges of R_i are identified with points of T, as in the second figure. Any arcs β_l that cross over α_i are positioned to lie in R_i. Finally, over each arc β_l put a square S_l, bent downward along its four edges so that these edges are identified with points of three strips R_i, R_j, and R_k as in the third figure; namely, two opposite edges of S_l are identified with short edges of R_j and R_k and the other two opposite edges of S_l are identified with two arcs crossing the interior of R_i. The knot K is now a subspace of X, but after we lift K up slightly into the complement of X, it becomes evident that X is a deformation retract of $\mathbb{R}^3 - K$.

1. Assuming this bit of geometry, show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator x_i for each strip R_i and one relation of the form $x_i x_j x_i^{-1} = x_k$ for each square S_l, where the indices are as in the figures above. [To get the correct signs it is helpful to use an orientation of K.]

2. Use this presentation to show that the abelianization of $\pi_1(\mathbb{R}^3 - K)$ is $\mathbb{Z}$.

Solution. 1. We shall take the assumption that X is homotopic to $\mathbb{R}^3 - K$, so computing $\pi_1(\mathbb{R}^3 - K)$ is actually computing $\pi_1(X)$.

First we compute the fundamental group of $T \cup R_1 \cup \cdots \cup R_i \cup \cdots$. Each R_i gives a nontrivial loop, so by Van Kampen's theorem $\pi_1(T \cup R_1 \cup \cdots \cup R_i \cup \cdots) = \mathbb{F}_n$ where $\mathbb{F}_n$ is the free group generated by n elements, and n is the number of such R_i's. Next, gluing S_l makes the loop $x_i x_j x_i^{-1} x_k^{-1}$ trivial by Van Kampen's theorem, so the fundamental group is as described.

2. After taking abelianisation, the braid relation $x_i x_j x_i^{-1} = x_k$ becomes $x_j = x_k$, hence the abelianisation is $\mathbb{Z}$. □

1.3 Covering Spaces

Exercise 1.3.1.

Solution. □

Exercise 1.3.2.

Solution. □

Exercise 1.3.3.

Solution. □

Exercise 1.3.4 (Exercise 1.3.4)**.** Construct a simply-connected covering space of the space $X \subseteq \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Solution. Let Z be countably infinitely many disjoint union of closed unit ball in $\mathbb{R}^3$ centered at z-axis, and connect their north pole N to the south pole S of the ball right above them. Let $Y = \partial Z$ as figure 1.1. Then let $p : Y \to X$, where p send the spheres to sphere, send the connecting segments to the diameter inside the sphere, and send north/south pole to the points where the diameter intersecting the sphere. Clearly Y is the covering space.

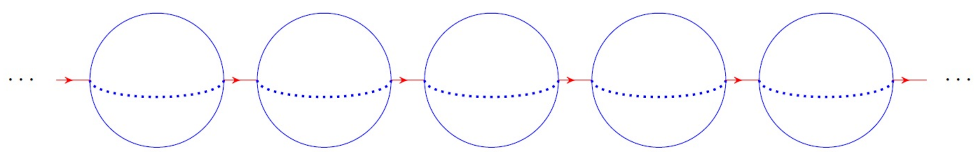


Figure 1.1: The Covering Space of X

For another case, we still have the same space but we have different map $p' : Y \to X'$. First label the connecting segments a, b consecutively, then p' send the spheres to sphere, send the connecting segments labeled as a to the half-circle inside the sphere and the connecting segments labeled as b to the half-circle outside the sphere, and send north/south pole to the points where the circle intersecting the sphere. □

Exercise 1.3.5 (Exercise 1.3.5)**.** Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ inside the square. Show that for every covering space $\tilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Solution. Denote X_m be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0, \frac{1}{m}] \times [0,1]$ together with the segments of the vertical lines $x = \frac{1}{m+1}, \frac{1}{m+2}, \frac{1}{m+3},$
$cdots$ inside the square. We first prove that for any integer m there is a homeomorphism $X \cong X_m$. Consider

$$\varphi : X \to X_m,$$

where φ maps a point $(\frac{1}{k}, t)$ to $(\frac{1}{k+m}, t)$ and φ maps a point $(x, 0)$ to $(\frac{1}{k+m} + (\frac{1}{k+m} - \frac{1}{k+m+1})\frac{x-\frac{1}{k+1}}{\frac{1}{k}-\frac{1}{k+1}}, 0)$ where $\frac{1}{k+1} \le x \le \frac{1}{k}$ for some integer k. Clearly φ and its inverse are piecewisely continuous on the intervals homeomorphic to I, hence it is a homeomorphism.

Suppose $Y \subset X$ is an open neighborhood of the left edge, then we have an open set U of $\mathbb{R}^2$ s.t. $Y = U \cap X$. For each point x in the left side, there is a ball $B_x(\delta_x)$ centered at x with radius δ_x s.t. $B_x(\delta_x) \subseteq U$, therefore we form an open covering of I. Since I is compact, there is a finite open covering $B_{x_1}(\delta_{x_1}), \cdots, B_{x_n}(\delta_{x_n})$. Let $\frac{1}{m} < \min\{\delta_{x_1}, \cdots, \delta_{x_n}\}$, then we know $X_m \subseteq U$, hence $X_m \subseteq Y$.

Suppose $p : \tilde{X} \to X$ is a covering space of X, then $p^{-1}(Y)$ has a connected piece homeomorphic to Y where $Y \subset X$ is an open neighborhood of the left edge. Thus we have a subspace homeomorphic to $X_m \cong X$. Therefore we have a nontrivial element $[\omega] \in \pi_1(\tilde{X})$ where $\omega : I \to I \cup [\frac{1}{2}, 1] \cup I \cup [\frac{1}{2}, 1]$. Otherwise $p_*([\omega])$ must be trivial in $\pi_1(X)$, which is a contradiction. □

Exercise 1.3.6.

Solution. ☐

Exercise 1.3.7 (Exercise 1.3.7)**.** Let Y be the quasi-circle, a closed subset of $\mathbb{R}^2$ consisting of a portion of the graph of $y = \sin \frac{1}{x}$, the segment $[-1, 1]$ in the y-axis, and an arc connecting these two pieces. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$ even though $\pi_1(Y) = 0$. Thus local path-connectedness is a necessary hypothesis in the lifting criterion.

Solution. Without loss of generality, assume the quotient map f sends the segment $[-1, 1]$ to the base point of S^1. Suppose there is a lift $\tilde{f}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"'] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1, \arrow[u, dashed, bend right, "q"'] \end{tikzcd}$$

then $\tilde{f}$ has to send the segment to (one point of) $\mathbb{Z}$. By the universal property of the quotient space, there is a map $q : S^1 \to \mathbb{R}$ s.t. $\tilde{f} = q \circ f$, thus $f = p \circ \tilde{f} = p \circ q \circ f$. Still by the universal property of quotient space (there exists only one map), $p \circ q = \mathrm{id}$, a contradiction. ☐

Exercise 1.3.8 (Exercise 1.3.8)**.** Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$, then $\tilde{X} \simeq \tilde{Y}$.

Solution. ☐

Exercise 1.3.9 (Exercise 1.3.9)**.** Show that if a path-connected, locally path-connected spaces X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

Solution. First we prove that if $f : X \to S^1$ is a map then the induced map $f_* : \pi_1(X) \to \pi_1(S^1)$ is the zero map. Suppose $n = |\pi_1(X)| > 0$ and suppose that $[\omega] \in \pi_1(X)$ is an element, then $[\omega]^n = 1 \in \pi_1(X)$. But $f_*([\omega]^n) = n \cdot f_*([\omega]) = 0 \in \pi_1(S^1)$, and the only element k in $\pi_1(S^1) = \mathbb{Z}$ s.t. $n \cdot k = 0$ is 0, so $f_*([\omega]) = 0$, i.e. $f_* : \pi_1(X) \to \pi_1(S^1)$ is the zero map.

By the lifting criterion, we have a lift $\tilde{f} : X \to \mathbb{R}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & S^1. \end{tikzcd}$$

Since $\mathbb{R}$ is contractible, by Exercise 0.0.6, $\tilde{f}$ is contractible. Thus we have a $F : X \times I \to \mathbb{R}$ s.t. $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = \text{constant}$. Therefore, $G := p \circ F$ is the homotopy from $\tilde{f}$ to a constant map. ☐

Exercise 1.3.10 (Exercise 1.3.10)**.** Find all the connected 2-sheeted and 3-sheeted covering space of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

Solution. By the textbook, the covering space $\tilde{X}$ of $X = S^1 \vee S^1$ is a 2-oriented graph. While the map $p : \tilde{X} \to X$ maps vertices to the vertex in X, hence there are n vertices if and only if the covering space is n-sheeted.

For $n = 2$, there are two possibilities, where in one case there is an edge that connects a vertex to itself, while in the other there is not. But the graph is connected, so there must be two edges connecting the two vertices. Hence there are only two different covering space, shown in the textbook Page 58, (1) and (2). And because of the symmetry of the two vertex, the orientation does not matter.

For $n = 3$, there are two possibilities, where in one case there is an edge that connects a vertex to itself, while in the other there is not. But the graph is connected, so there must be two edges connecting some the two vertices. Thus there must be a vertex without any single edge connecting to itself. So there are also two cases, where in one case there is an edge that connects a vertex to itself, while in the other there is not (corresponding to the (3),(4) and (5),(6) in the table on Page 58 respectively). There is also the consideration of orientation of the edges. Since for each vertex, there are exactly two paths labeled as a (or b), and exactly one coming into the vertex. So for each case, there are two nonisomorphic covering space, which are (3),(4) and (5),(6) in the table on Page 58. ☐

Exercise 1.3.11.

Solution. ☐

Exercise 1.3.12.

Solution. ☐

Exercise 1.3.13.

Solution. ☐

Exercise 1.3.14.

Solution. ☐

Exercise 1.3.15.

Solution. ☐

Exercise 1.3.16 (Exercise 1.3.16)**.** Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if Z is locally path-connected. Show that this covering space is normal if $X \to Z$ is normal.

Solution. Suppose the given maps are $p : X \to Y$ and $q : Y \to Z$. Since q and $q \circ p$ are covering spaces, by definition for each $y \in Y$, there is an open set $U \subseteq Z$ containing $q(x)$ s.t. both $q^{-1}(U)$ and $(q \circ p)^{-1}(U) = p^{-1}(q^{-1}(U))$ are disjoint union of spaces that are homeomorphic to U via q and $q \circ p$ respectively. Since Z is locally path-connected, one can take the neighborhood U small enough so that U is path-connected.

Since q is a covering space, there is an open subset $V \subseteq q^{-1}(U)$ s.t. $q|_V$ gives a homeomorphism between spaces. Therefore one has a commutative diagram

$$\begin{tikzcd}
& p^{-1}(V) \arrow[r, hook] \arrow[d, "p|_{p^{-1}(V)}"] & X \arrow[d, "p"] \\
& V \arrow[r, hook] \arrow[d, "q|_V"] & Y \arrow[d, "q"] \\
U \arrow[r, equal] \arrow[ur, "(q|_V)^{-1}"] \arrow[uur, dashed, bend left, "f_x|_{p^{-1}(V)}"] & U \arrow[r, hook, "i"'] & Z.
\end{tikzcd}$$

For any $x \in p^{-1}(V) \subseteq X$ s.t. $p(x) = y$, there is a lift $f_x : U \to X$ s.t.

$$i = q \circ p \circ f_x : U \to X \to Y \to Z$$

where $i : U \hookrightarrow X$ is the embedding, since $X \xrightarrow{p} Y \xrightarrow{q} Z$ is a covering space. Thus

$$p^{-1}(V) \subseteq p^{-1}(q^{-1}(q(V))) \subseteq p^{-1}(q^{-1}(U)),$$

which is a disjoint union of open sets, say $\coprod_j W_j$. There is one such $j = j_0$ s.t. $x \in W_{j_0}$. Because U is path-connected (hence connected), there is a unique lift $U \to X$ to $q \circ p$, which is f_x as constructed. This means $p|_{W_{j_0}}$ is a homeomorphism. Furthermore, since x is arbitrary in the fibre of any point $y \in Y$, hence $p^{-1}(V)$ is a disjoint union of open subsets in X homeomorphic to V via p.

For the second part, one has

$$\begin{tikzcd}
X \arrow[d, two heads, "p"] & \pi_1(X) \arrow[d, hook] \\
Y \arrow[d, two heads, "q"] & \pi_1(Y) \arrow[d, hook] \\
Z & \pi_1(Z),
\end{tikzcd}$$

notice that since $X \to Z$ is normal, the normalisor of $\pi_1(X)$ in $\pi_1(Z)$ is the whole group, therefore the normalisor of $\pi_1(X)$ in $\pi_1(Y)$ is the intersection $N_{\pi_1(X)}(\pi_1(Z)) \cap \pi_1(Y)$, which is $\pi_1(Y)$. Hence $X \to Y$ is normal if X is path-connected. Otherwise, notice that

1. Permutation of homeomorphic path-connected components of X is a deck transformation.
2. A deck transformation can only map a path-connected component to a component homeomorphic to it.

Therefore for any two points in the fibre of $y \in Y$, via a permutation of homeomorphic path-connected components composed with a deck transformation within the path-connected component (shown before), one gets such a deck transformation. $\square$

Exercise 1.3.17 (Exercise 1.3.17)**.** Given a group G and a normal subgroup N, show that there exists a normal covering space $\tilde{X} \to X$ with $\pi_1(X) = G$, $\pi_1(\tilde{X}) = N$, and deck transformation group $G(\tilde{X}) = G/N$.

Solution. By Corollary 1.28, there is a 2-dimensional cell complex X s.t. $\pi_1(X) = G$. By Prop 1.36, we have a covering space $p : \tilde{X} \to X$ s.t. $p_*(\pi_1(\tilde{X})) = N$. Since N is a normal subgroup, by Prop 1.39, we have $\tilde{X}$ is a normal covering space and the deck transformation group is G/N. $\square$

Exercise 1.3.18 (Exercise 1.3.18)**.** For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\tilde{X} \to X$ abelian if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Solution. Since the space X is path-connected, locally path-connected, and semilocally simply-connected space, there is a universal covering space $X_0 \to X$, whose deck transformation is exactly $\pi_1(X)$.

Consider the normal group $[\pi_1(X), \pi_1(X)]$, By Prop 1.36. there is a covering space $p : X_{[\pi_1(X),\pi_1(X)]} \to X$ s.t. $p_*(X_{[\pi_1(X),\pi_1(X)]}) = [\pi_1(X), \pi_1(X)]$. By Prop 1.39., $p : X_{[\pi_1(X),\pi_1(X)]} \to X$ is normal and the deck transformation group is

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] = \pi_1(X)_{\text{ab}}.$$

To show that $\tilde{X} := X_{[\pi_1(X),\pi_1(X)]}$ is universal among all abelian covering spaces, consider another abelian covering space $X_K \to X$, with deck transformation group

$$\pi_1(X)/K = \pi_1(X)/p(\pi_1(X_K))$$

(where $\pi_1(X)/K$ is only a notation of the group). Since $\pi_1(X)/p(\pi_1(X_K))$ is abelian, there is a unique induced group homomorphism g making the diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[r, equal] \arrow[d] & \pi_1(X) \arrow[d] \\
\pi_1(X)_{\text{ab}} \arrow[r, dashed, "g"] & \pi_1(X)/p(\pi_1(X_K))
\end{tikzcd}$$

commute, by the universal property of the abelianisation. The diagram implies that

$$p_*(X_{[\pi_1(X),\pi_1(X)]}) = [\pi_1(X), \pi_1(X)] \subseteq \pi_1(X_K),$$

thus there is a "lift"

$$\begin{tikzcd}
 & X_K \arrow[d] \\
\tilde{X} \arrow[ur, dashed] \arrow[r] & X
\end{tikzcd}$$

by Prop 1.33. and the "lift" is unique by 1.34. since X is path-connected. The uniqueness comes from abstract nonsense.

By discussion above, the universal abelian covering is the quotient space of the derived group acting on the universal covering space. One knows that the universal covering space of $S^1 \vee S^1$ (resp. $S^1 \vee S^1 \vee S^1$) is the tree indexed by the free group $\mathbb{F}_2$ (resp. $\mathbb{F}_3$). Thus the universal abelian covering space is a quotient space of the tree identifying the corresponding vertices (hence the edges) in the normal subgroup of the commutator subgroup. These happen to give the lattice grids in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. $\square$

Exercise 1.3.19 (Exercise 1.3.19)**.** Use the preceding problem to show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of n copies of $\mathbb{Z}$) iff $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations. Show that such a covering space in $\mathbb{R}^3$ exists iff there is an embedding of M_g in the 3 torus $T^3 = S^1 \times S^1 \times S^1$ such that the induced map $\pi_1(M_g) \to \pi_1(T^3)$ is surjective.

Solution. By the discussion in Exercise 1.3.18, notice that

$$\pi_1(M_g) = \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

and its abelianisation is $\mathbb{Z}^{2g}$. Thus, the universal abelian covering space $\tilde{X} \to X$ has deck transformation group $\mathbb{Z}^{2g}$. By the universal property, for any abelian covering $X' \to X$, there is a normal covering $\tilde{X} \to X'$ (by Exercise 1.3.16)

$$
\begin{tikzcd}
\tilde{X} \arrow[d, two heads] & \pi_1(\tilde{X}) \arrow[d, hook] & G(\tilde{X}/\tilde{X}) = \{e\} \\
X' \arrow[d, two heads] & \pi_1(X') \arrow[d, hook] & G(\tilde{X}/X') = \pi_1(X')/\pi_1(\tilde{X}) \arrow[u, two heads] \\
X & \pi_1(X) & G(\tilde{X}/X) = \pi_1(X)/\pi_1(\tilde{X}) \arrow[u, two heads]
\end{tikzcd}
$$

inducing the surjections among the (relative) deck transformation groups by Prop 1.39. Since $G(\tilde{X}/X) = \pi_1(X)/\pi_1(\tilde{X})$ has $2g$ generators, the number of generators n in $G(\tilde{X}/X') = \pi_1(X')/\pi_1(\tilde{X})$ is less than $2g$. Conversely, for $n \leq 2g$, there is a surjection $\mathbb{Z}^{2g} = (\pi_1(M_g))_{\text{ab}} \to \mathbb{Z}$ so one can construct an abelian covering space with deck transformation group $\mathbb{Z}^n$ as it is constructed in Exercise 1.3.18.

Before constructing the covering space, we first show that such a covering space in $\mathbb{R}^3$ *with deck transformations being $\mathbb{Z}^3$ translations* exists iff there is an embedding of M_g in the 3 torus $T^3 = S^1 \times S^1 \times S^1$ such that the induced map $\pi_1(M_g) \to \pi_1(T^3)$ is surjective. On the one hand, if there is such an embedding,

$$
\begin{tikzcd}
\tilde{M}_g \arrow[r, dashed, "\tilde{i}"] \arrow[d, two heads, "p"] & \mathbb{R}^3 \arrow[d, "q"] \\
M_g \arrow[r, hook, "i"] & T^3
\end{tikzcd}
$$

then the composition $\tilde{M}_g \to M_g \to T^3$ induces a composition of groups $\pi_1(\tilde{M}_g) \to \pi_1(M_g) \to \pi_1(T^3)$, where the image of $\pi_1(\tilde{M}_g)$ is the commutator subgroup, hence it maps to 0 since $\pi_1(T^3) = \mathbb{Z}^3$ is abelian. Therefore there is a lift $\tilde{M}_g \dashrightarrow \mathbb{R}^3$ making the diagram commute. To show that this lift is injective, suppose there are two points $\tilde{x}_0, \tilde{x}_1$ are mapped to be the same point in $\mathbb{R}^3$ hence the same point in T^3. Since $M_g \to T^3$ is injective, and the diagram is commutative, $\tilde{x}_0, \tilde{x}_1$ are mapped to the same point in M_g, they are in the same fibre. For any path $\gamma : I \to \tilde{M}_g$ connecting these two points provided they are different, then by the unique lifting, $[p \circ \gamma] \notin p_*(\pi_1(\tilde{M}_g))$, implying $[i \circ p \circ \gamma] \in \pi_1(T^3)$ is not trivial. However, $[i \circ p \circ \gamma] = [q \circ \tilde{i} \circ \gamma]$ is trivial since $\pi_1(\mathbb{R}^3)$ being trivial, a contradiction. So $\tilde{i}$ is an embedding. Given any deck transformation σ, the composition

$$\tilde{M}_g \xrightarrow{\sigma} \tilde{M}_g \hookrightarrow \mathbb{R}^3$$

gives the same lift up to a choice of base point. Therefore

$$
\begin{tikzcd}
\tilde{M}_g \arrow[r, dashed] \arrow[d, "\sigma"] & \mathbb{R}^3 \arrow[dd, bend left] \\
\tilde{M}_g \arrow[r, dashed] \arrow[d, two heads, "p"] & \mathbb{R}^3 \arrow[d] \\
M_g \arrow[r, hook] & T^3
\end{tikzcd}
$$

is commutative. The way to find the deck transformations is to correspond a loop in $\pi_1(M_g)$ using the lift to give the deck transformation. Since all covering spaces are normal, one can use the correspondence

$$\mathrm{Gal}(\tilde{M}_g/M_g) \cong \pi_1(M_g)/p_*(\pi_1(\tilde{M}_g))$$

to describe a deck transformation. For any deck transformation σ whose image in $i_*\pi_1(M_g) \subseteq \pi_1(T^3)$ is not zero, then its image in $i_*\pi_1(M_g) \subseteq \pi_1(T^3)$ corresponds to a deck transformation $\tilde{\sigma} : \mathbb{R}^3 \to \mathbb{R}^3$. However, all diagrams given are commutative and the action of σ is constructed via the action on fibres, hence there is actually a diagram

$$\begin{tikzcd} \tilde{M}_g \arrow[r, dashed] \arrow[d, "\sigma"] & \mathbb{R}^3 \arrow[d, "\tilde{\sigma}"'] \arrow[dd, bend left] \\ \tilde{M}_g \arrow[r, dashed] \arrow[d, "p"] & \mathbb{R}^3 \arrow[d] \\ M_g \arrow[r, hook] & T^3. \end{tikzcd}$$

This means all deck transformations of $\mathrm{Gal}(\tilde{M}_g/M_g)$ are transformations in $\mathrm{Gal}(\mathbb{R}^3/\mathbb{Z}^3) = \mathbb{Z}^3$. Also because $\pi_1(M_g) \to \pi_1(T^3)$ is surjective, the transformations subgroup is the whole group.

On the other hand, suppose there is subspace

$$\tilde{M}_g \hookrightarrow \mathbb{R}^3$$

with that all deck transformations of $\tilde{M}_g$ can be extended to be a unique action of $\mathbb{Z}^3$ on $\mathbb{R}^3$. Then the embedding induces a continuous map

$$M_g \cong \tilde{M}_g/G = \tilde{M}_g/\mathbb{Z}^3 \to \mathbb{R}^3/\mathbb{Z}^3 = T^3$$

by the group action. This map is injective because the map is quotienting out the same group action. Notice that

$$\pi_1(M_g)/p_*(\pi_1(\tilde{M}_g)) \cong \mathrm{Gal}(\tilde{M}_g/M_g) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{R}^3/T^3) = \pi_1(T^3)/p_*(\pi_1(\mathbb{R}^3)) = \pi_1(T^3),$$

thus $\pi_1(T^3)$ is a quotient group of $\pi_1(M_g)$ via the identification of deck transformations, hence via the inclusion $\pi_1(T^3) \to \pi_1(M_g)$ is surjective.

Finally we want to construct one embedding for $g \geq 3$. For the case of $g = 3$, notice that M_g *is the boundary of a tubular neighbourhood of* $S^1 \vee S^1 \vee S^1$ *embedded in* $\mathbb{R}^3$. We have done the case of $S^1 \vee S^1 \vee S^1$ for a abelian covering space embedded in $\mathbb{R}^3$ with *three generators commuting*. With "these" three generators commuting for M_3, the abelian covering space should be the boundary of a tubular neighbourhood of the abelian covering space of $S^1 \vee S^1 \vee S^1$ embedded in $\mathbb{R}^3$, with the deck transformations being the translations by the lattice. For the case $g > 3$, we should not have more deck transformations as described above, hence other "genus" should be trivial under all the deck transformations, where the way to construct is to put $g - 3$ handles to each unit of the deck transformation action. □

Exercise 1.3.20 (Exercise 1.3.23)**.** Show that if a group acts freely and properly discountinuously on a Hausdorff space X, then the action is a covering space action. (Here 'properly discountinuously' means that each $x \in X$ has a neighborhood U such that $\{g \in G \mid U \cap gU \neq \emptyset\}$ is finite). In particular, a free action on a finite group on a Hausdorff space is a covering space action.

Solution. First we shall prove that for each $x \in X$, there is a neighborhood V s.t. $V \cap gV \neq \emptyset$ implies that $g = e$. By the proper-discountinuity, there are finitely many open neighborhood $g_1, \cdots, g_n$ s.t. all elements $g \in G - \{e\}$ but them satisfy that

$$gU \cap U = \emptyset.$$

Since X is Hausdorff, for each $1 \leq i \leq n$, there are open sets $U_i \ni x$ and $V_i \ni g_i \cdot x$ s.t. $U_i \cap V_i = \emptyset$. Take

$$V := U \cap U_1 \cap \cdots \cap U_n \cap g_1^{-1} V_1 \cap \cdots \cap g_n^{-1} V_n,$$

then $V \subseteq U_i$ and $g_i V \subseteq V_i$, hence V is the open neighborhood we want.

Then we shall prove that the action is a covering space action. For any two g_1, g_2 s.t. $g_1 V \cap g_2 V \neq \emptyset$, then $V \cap g_2^{-1} g_1 V \neq \emptyset$. Thus $g_2^{-1} g_1 = e$ and $g_1 = g_2$. By Prop 1.40, this is a covering space action. □

Exercise 1.3.21 (Exercise 1.3.25)**.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\varphi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$. Show this action is a covering space action and compute $\pi_1(X/\mathbb{Z})$. Show the orbit space $X/\mathbb{Z}$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis.

Solution. First to show that this is a covering space action, which is to find an open neighbourhood U s.t. $gU \cap U \neq \emptyset$ if and only if $g = e$. For any $(x_0, y_0) \in \mathbb{R}^2 - \{(0,0)\}$, W.L.O.G. one can assume $x_0 \neq 0$ then take $U := \{(x, y) \mid x_0 - \frac{|x_0|}{4} < x < x_0 + \frac{|x_0|}{4}\}$, then

$$n \cdot U = \left\{ (x,y) \middle| x_0 - \frac{|x_0|}{4} < 2^{-n}x < x_0 + \frac{|x_0|}{4} \right\} = \left\{ (x,y) \middle| 2^n x_0 - 2^{n-2}|x_0| < x < 2^n x_0 + 2^{n-2}|x_0| \right\},$$

thus $n \cdot U \cap U \neq \emptyset$ iff $n = 0$.

There is an algebraic way of computing $\pi_1(X/\mathbb{Z})$. By Prop 1.40, X is a covering space of $X/\mathbb{Z}$ with the deck transformation group $\mathbb{Z}$, such that

$$\frac{\pi_1(X/\mathbb{Z})}{p_*(\pi_1(X))} \cong \mathbb{Z}.$$

Therefore one has a S.E.S. of groups

$$0 \to \pi_1(X) \cong \mathbb{Z} \to \pi_1(X/\mathbb{Z}) \to \frac{\pi_1(X/\mathbb{Z})}{p_*(\pi_1(X))} \cong \mathbb{Z} \to 0$$

where the first one is injective since it is induced by the covering map. Therefore $\pi_1(X/\mathbb{Z})$ has two generators, denoted by $[\omega], [\gamma]$ respectively, where $[\omega]$ comes from the image of $\pi_1(X)$ under p_* and $[\gamma]$ is mapped to be the generator of $\frac{\pi_1(X/\mathbb{Z})}{p_*(\pi_1(X))}$. On the one hand, one can take $\omega : I \to X, t \mapsto \exp 2\pi i t$ then this gives the generator $[\omega]$. On the other hand, $[\gamma]$ is a class in $\pi_1(X/\mathbb{Z})$ but not a class in $p_*(\pi_1(X))$, then if one takes a representative $\gamma : I \to X/\mathbb{Z}$, it is a loop in $X/\mathbb{Z}$. By the path lifting property (Prop 1.30), there is a (unique) lift $\tilde{\gamma} : I \to X$. However, $[\gamma] \notin p_*(\pi_1(X))$ means that the lift cannot be $[\omega^n]$ for some nonzero n, therefore $\tilde{\gamma}$ is a contractible path in X. Hence there is a homotopy in X s.t.

$$\tilde{\gamma} \cdot \omega \tilde{\gamma}^{-1} \simeq \omega,$$

therefore it induces a homotopy in $X/\mathbb{Z}$ s.t. $[\omega]$ commutes with $[\gamma]$. In conclusion, $\pi_1(X/\mathbb{Z}) \cong \mathbb{Z}^2$.

To show that the orbit space is not Hausdorff, it suffices to find a sequence of points with two different limit points. Consider the sequences $\{(1, 2^n)\} \sim \{(2^n, 1)\}$, where they are the same sequence under the group action. However, the first sequence converges to $(1, 0)$ as $n \to -\infty$ while the second one converges to $(0, 1)$. Two limit points are not in the same equivalence.

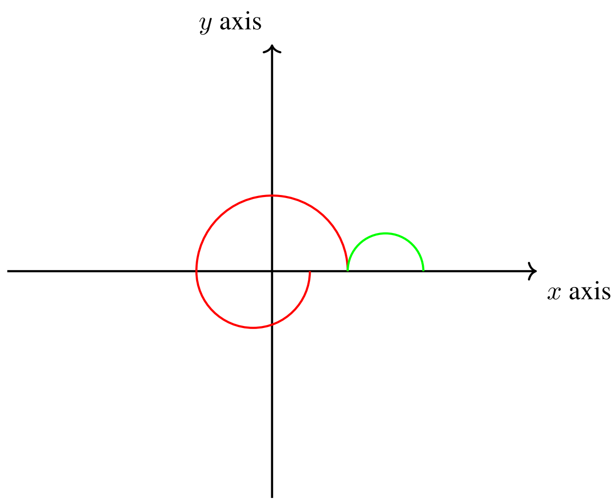
Finally consider the curve

$$\{(x, y) \mid xy = c, x > 0\}$$

for some constant $c \in \mathbb{R}_+$, then under the group action this becomes an S^1. Therefore the first quadrant is one piece of $S^1 \times \mathbb{R}$, where the four cylinder are given by the quadrants, with axis giving four S^1's. $\square$

Remark. There is a way of visualising the fundamental group of $X/\mathbb{Z}$.

Take $x_0 = (1, 0)$ to be the base point of X and its image under the quotient to be the base point of $X/\mathbb{Z}$. Thus both of the paths below



are loops in $X/\mathbb{Z}$. All these are homotopic to $[\gamma]$ and the unit circle is ω.

Appendix 1.A Trees

Exercise 1.A.1 (Exercise 1.A.3)**.** For a finite graph X, define the Euler characteristic $\chi(X)$ to be the number of the vertices minus the number of edges. Show that $\chi(X) = 1$ if X is a tree, and that the rank (number of elements in a basis) of $\pi_1(X)$ is $1 - \chi(X)$ if X is connected.

Solution. By the discussion in the textbook, we know a graph is a tree if and only if it is simply connected. We can also prove that a connected graph X is a tree if and only if for any two vertices v_1, v_2, there exists exactly one path connecting them. If X is a tree, then there is ne guaranteed by the connectedness. If there are at least two pathes, then we find a loop, a contradiction. Conversely, suppose X has the property, then it is connected. If there is a loop, then we find two different pathes connecting two points, a contradiction.

Back to the problem, we use induction to prove it. When there is 1 vertex, there is no edges, hence $\chi(X) = 1$. Suppose there are n vertices. If we can find a vertex v s.t. there is only one different vertex connecting to it (we can call v a leaf.), then if we delete v and the unique edge connecting to it, the remaining space is still a graph with $n-1$ vertices. By induction hypothesis, there are $n-2$ edges in the new graph, hence $\chi(X) = (n-1+1) - (n-2+1) = 1$.

If we cannot find such a leaf, then starting from a vertex, we can find a path with infinitely many edges. But there are only finitely many vertices, hence the loop intersects with itself. Therefore, we have a loop, which is a contradiction.

For the second part, consider the maximal tree of X. □

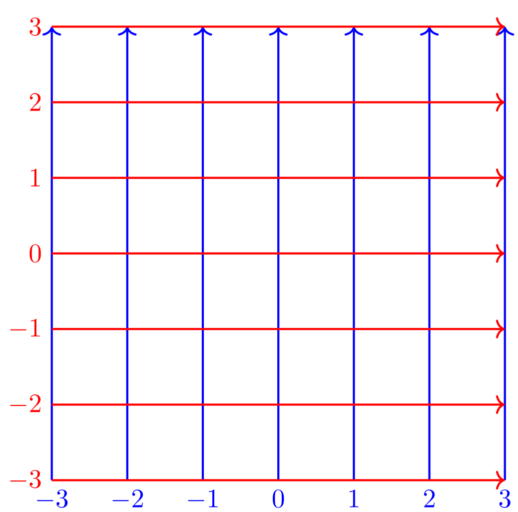
Exercise 1.A.2 (Exercise 1.A.4)**.** If X is a finite graph and Y is a subgraph homeomorphic to S^1 and containing the basepoint x_0, show that $\pi_1(X, x_0)$ has a basis in which one element is represented by the loop Y.

Solution. Suppose Y consists of the vertices $x_0, x_1, \cdots, x_n$, since $Y \cong S^1$, all the edges are $x_0x_1, \cdots, x_ix_{i+1}, \cdots, x_nx_0$. Thus, if S is the maximal tree of Y, then S consists of $n+1$ vertices and n edges by 1.A.3.. Hence there is exactly 1 edge in $Y - S$, which is also the only edge in Y/S.

Suppose T is the maximal tree in X, then we know T is contractible and $\pi_1(X, x_0) = \pi_1(X/T, x_0)$, where X/T is the space where we contract T to the point x_0. Thus $Y \cap T$ is also the maximal tree of T. By previous discussion, $Y/(Y \cap T)$ consists of x_0 and only one edge out of $Y \cap T$, hence by Prop. 1.A.2, we have a basis corresponding to $[f_\alpha]$ where f_α are all the edges not in T, therefore the element corresponding to the edge in $Y/(Y \cap T)$ is represented by the loop Y. □

Exercise 1.A.3 (Exercise 1.A.6)**.** Let F be a free group on two generators and let F' be its commutator subgroup. Find a set of free generators for F' by considering the covering space of the graph $S^1 \vee S^1$ corresponding to F'.

Solution. The commutator group F' is generated by $aba^{-1}b^{-1}$, hence the graph is



where we label the red edges by a and the blue edges by b. It's a 2-graph, so it is a covering space of $S^1 \vee S^1$. The only relation we can find is $aba^{-1}b^{-1} = 1$, hence it is the covering space corresponding to F'. By 1.A.4., we know F' is a free group. One constructs its maximal tree T, consisting of all vertices, all vertical (blue) lines and the x-axis. Since T is contractible, the generators of F' can be corresponded with all blue edges in $X - T$. □

Appendix 1.B $K(G,1)$

Exercise 1.B.1 (Exercise 1.B.2)**.** Let X be a connected CW complex and G a group such that every homomorphism $\pi_1(X) \to G$ is trivial. Show that every map $X \to K(G,1)$ is nullhomotopic.

Solution. Given any continuous map $f : X \to K(G,1)$, there is an induced map $0 = f_* : \pi_1(X, x_0) \to \pi_1(K(G,1)) = G$. By Prop 1.B.9., any group homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(K(G,1)) = G$$

is induced uniquely up to a homotopy. Since the constant map const also induces $0 : \pi_1(X, x_0) \to \pi_1(K(G,1)) = G$, these two maps are homotopic relative to the base point. □

Remark. If one considers Prop 1.B.8. in the sense of homotopy theory, this is a simple implication of 1.B.9. and the Whitehead theorem. This problem has the same flavor, which is that all higher homotopy groups of $K(G,1)$ are trivial, and the induced map on the fundamental groups is also trivial, hence all maps are weak homotopy equivalences. In good cases, weak homotopy equivalences are actual homotopy equivalences.

Chapter 2

Homology

2.1 Simplicial and Singular Homology

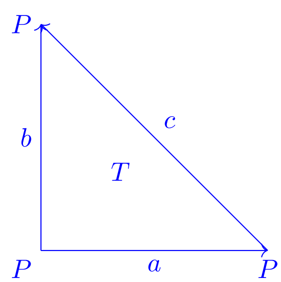
Remark. I'm a homological-algebraic person, so I'll omit all the basic definitions/results/problems from homological algebras.

Exercise 2.1.1 (Exercise 2.1.3)**.** Construct a Δ-complex structure on $\mathbb{RP}^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.

Solution. The construction of $\mathbb{RP}^n$ is to identify the antipodal points of S^n. We will construct the Δ-complex by induction. First, for S^1, pick two antipodal points, and denote them by v_0, v_1, and denote by $a = [v_0, v_1], b = [v_0, v_1]$ the two segments. Hence $\mathbb{RP}^1$ is to identify $v_0 = v_1$ and $a = -b$. Then we consider $\mathbb{RP}^n$. Since the interior of Δ^n is homeomorphic to the interior of D^n, where D^n is the union of two Δ^n intersecting along an $n-1$-face. We construct the Δ-complex structure on S^n as follows: the equator of S^n is homeomorphic to S^{n-1}, on S^{n-1} we construct the Δ-complex by induction hypothesis so that the quotient can be applied. The complement of the equator is the disjoint union of two $D^n - \partial D^n$, which is homeomorphic to the interior of Δ^n so we have two Δ^n whose boundary are both the equator with the same orientation. Thus we have the Δ-complex structure on S^n and the Δ-complex structure on $\mathbb{RP}^n$ is derived by quotient. $\square$

Exercise 2.1.2 (Exercise 2.1.4)**.** Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

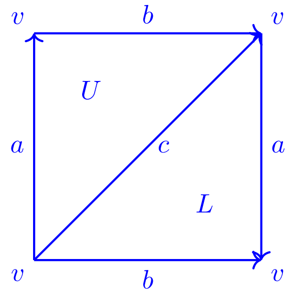
Solution. Denote the 2-cell as T and the three edges with orientations as a, b, c, and denote the only point P as follows.



Thus we know that the nontrivial Δ-complex groups are $C_2 = \mathbb{Z} = \mathbb{Z}[T]$, $C_1 = \mathbb{Z}^3 = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c]$ and $C_0 = \mathbb{Z} = \mathbb{Z}[P]$. Notice that $\partial_2(T) = a - b + c$ and $\partial_1(a) = \partial_1(b) = \partial_1(c) = P - P = 0$, hence $\mathrm{Ker}\ \partial_2 = 0$, $\mathrm{Ker}\ \partial_1 = \mathbb{Z}^3 = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c]$ and $\mathrm{Im}\ \partial_1 = \mathbb{Z}[a - b + c] = \mathbb{Z}$. Thus we know that $H_2^\Delta = 0$, $H_1^\Delta = \mathbb{Z}^2$ and $H_0^\Delta = \mathbb{Z}$. $\square$

Exercise 2.1.3 (Exercise 2.1.5)**.** Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Solution. We denote the generators of the simplicial groups as follows.



Hence we know the Δ-complex groups are $C_2 = \mathbb{Z}^2 = \mathbb{Z}[U] \oplus \mathbb{Z}[L]$, $C_1 = \mathbb{Z}^3 = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c]$ and $C_0 = \mathbb{Z} = \mathbb{Z}[v]$. Notice that $\partial_2(U) = a+b-c$, $\partial_2(L) = a-b+c$ and $\partial_1(a) = \partial_1(b) = \partial_1(c) = P - P = 0$ and hence $\text{Ker}\ \partial_1 = \mathbb{Z}^3$. Suppose there are integers k, l s.t. $kU + lL \in \text{Ker}\ \partial_2$, then $k(a+b-c)+l(a-b+c) = 0$. By the linear independence, $k = l = 0$. Hence $\text{Ker}\ \partial_2 = 0$. Suppose there are integers u, v, w s.t. $ua+vb+wc \in \text{Im}\ \partial_2$, then we have integers k, l s.t. $ua+vb+wc = \partial_2(kU+lL) = (k+l)a+(k-l)b+(l-k)c$. Therefore $H_2^{\Delta} = 0$, $H_1^{\Delta} = (\mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c])/\{ua+vb-vc \mid (u,v,w) \in \mathbb{Z}^3\} = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ and $H_0^{\Delta} = \mathbb{Z}$. $\square$

Exercise 2.1.4 (Exercise 2.1.8)**.** Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \cdots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}., subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, 0, \mathbb{Z}$.

Solution. We denote the top and the bottom face of T_i by F_i, denote the vertical face between T_i and T_{i+1} by H_i. Then we denote the top and bottom vertex by w and the vertices on the rim $v = v_i$, where the outer vertex of H_i are v_i. Finally we denote the edge $[v_i, v_{i+1}]$ by b, the edge $[w, v_i]$ by a_i and the only vertical segment h. Thus

$$C_3 = \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}[T_i]$$

$$C_2 = \mathbb{Z}^{2n} = \bigoplus_{i=1}^n \mathbb{Z}[F_i] \oplus \bigoplus_{i=1}^n \mathbb{Z}[H_i]$$

$$C_1 = \mathbb{Z}^{n+2} = \mathbb{Z}[h] \oplus \bigoplus_{i=1}^n \mathbb{Z}[a_i] \oplus \mathbb{Z}[b]$$

$$C_0 = \mathbb{Z}^2 = \mathbb{Z}[v] \oplus \mathbb{Z}[w]$$

where $T_i = [w, w, v_i, v_{i+1}]$, $F_i = [w, v_i, v_{i+1}]$ and $H_i = [w, w, v_i]$. And

$$\begin{aligned}
&\partial(T_i) = H_{i+1} - H_i + F_{i+1} - F_i;\\
&\partial(F_i) = b - a_{i+1} + a_i, \partial(H_i) = h + a_i - a_{i+1};\\
&\partial_1(h) = 0, \partial(a_i) = v_i - w, \partial(b) = 0.
\end{aligned}$$

Thus

1. $\text{Ker}\ \partial_3 = \langle T_1 + T_2 + \cdots + T_n \rangle = \mathbb{Z}$, hence $H_3 = \mathbb{Z}$.
2. $\text{Ker}\ \partial_2 = 0$, so $H_2 = 0$.
3. X is connected and path-connected, so $H_0 = \mathbb{Z}$.
4. Clearly $\mathbb{Z}[h] \subseteq \text{Ker}\ \partial_1$. Suppose $\sum_i k_i a_i + kb$ is an element in the kernel of ∂_2, then

$$0 = \sum_{i=1}^n k_i(v_i - w) = \left(\sum_{i=1}^n k_i\right)(v - w).$$

By the linear independence, we must have $\sum_{i=1}^n k_i = 0$. On the other hand

$$\partial_3\left(\sum_{i=1}^n k_i F_i + l_i H_i\right) = \sum_{i=1}^n (k_i(a_i - a_{i+1} + b) + l_i(a_i - a_{i+1} + h)).$$

Thus

$$\begin{aligned} H_1 = \text{Ker } \partial_1/\text{Im } \partial_2 &= \frac{\{\sum_i k_i a_i + kb + lh \in C_1 \mid \sum_{i=1}^n k_i = 0\}}{\langle a_i - a_{i+1} + b, a_i - a_{i+1} + h\rangle} \\ &= \frac{\langle a_i - a_{i+1}, b, h\rangle}{\langle a_i - a_{i+1} + h, b - h\rangle} \\ &= \frac{\langle a_i - a_{i+1}, b, h\rangle}{\langle a_i - a_{i+1} + h, b - h, nh\rangle} \\ &= \mathbb{Z}/n\mathbb{Z} \end{aligned}$$

as desired by the fundamental theorem of f.g. abelian groups.

□

Exercise 2.1.5 (Exercise 2.1.9)**.** Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k-simplex for each $k \leq n$.

Solution. We know that $C_n(X) = C_{n-1}(X) = \cdots = C_1(X) = C_0(X) = \mathbb{Z}$. Suppose the generator of $C_i(X)$ is Δ^i, then

$$\partial_i \Delta^i = \partial_i([v_0, \cdots, v_i]) = \sum_{j=0}^{i} (-1)^j [v_0, \cdots, \hat{v_j}, \cdots, v_i].$$

By the gluing, we know that $[v_0, \cdots, \hat{v_j}, \cdots, v_i]$ are all identical, (here we assume the orientation is given when identifying) hence if i is odd

$$\partial_i \Delta^i = \sum_{j=0}^{i} (-1)^j [v_0, \cdots, \hat{v_j}, \cdots, v_i] = 0$$

and if i is even

$$\partial_i \Delta^i = \sum_{j=0}^{i} (-1)^j [v_0, \cdots, \hat{v_j}, \cdots, v_i] = \Delta^i.$$

Therefore we have the sequence

$$\cdots C^3 = \mathbb{Z} \xrightarrow{0} C^2 = \mathbb{Z} \xrightarrow{\text{id}} C^1 = \mathbb{Z} \xrightarrow{0} C^0 = \mathbb{Z},$$

hence

$$H_i(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \text{ when } n \text{ is odd;} \\ 0 & \text{otherwise} \end{cases}.$$

□

Exercise 2.1.6 (Exercise 2.1.11)**.** Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subseteq X$ is injective.

Solution. This could be done by abstract nonsense. For A a retract of X, there is a pair of maps

$$i : A \leftrightarrows X : r$$

s.t. $r \circ i = \text{id}_A$. By the functoriality of H_* (Prop 2.9 and Thm 2.10), □

Exercise 2.1.7 (Exercise 2.1.13)**.** Verify that $f \simeq g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.

Solution. Thm 2.10 shows that □

Exercise 2.1.8 (Exercise 2.1.14)**.** Determine whether there exists a short exact sequence $0 \to \mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z} \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}/p^m\mathbb{Z} \to A \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ with p prime. What about the case of short exact sequence $0 \to \mathbb{Z} \to A \to \mathbb{Z}/n\mathbb{Z} \to 0$?

Solution. We know that there is an element of order 4 in $\mathbb{Z}/4\mathbb{Z}$ and since the map $\mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ is injective, thus the image of $1 \in \mathbb{Z}/4\mathbb{Z}$ must be $(2,0)$ or $(2,1)$ because there are the only elements of order 4. But the element $(1,0) + \langle (2,1) \rangle$ is of order 4, hence there is S.E.S.

$$0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{1 \mapsto (2,1)} (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{(1,0) \mapsto 1} \mathbb{Z}/4\mathbb{Z} \to 0$$

Similar to the argument above, we have an element of order p^m since the map $\mathbb{Z}/p^m\mathbb{Z} \to A$ is injective. Since $\mathbb{Z}/p^m\mathbb{Z}$ and $\mathbb{Z}/p^n\mathbb{Z}$ are all cyclic groups, we know A is generated by at most 2 generators. If A have rank greater or equal than 1, then $\mathbb{Z}/p^m\mathbb{Z}$ must be mapped into the torsion part. But the kernal of the map $A \to \mathbb{Z}/p^n\mathbb{Z}$ is $\mathbb{Z}/p^m$, a contradiction. By the fundamental theorem of finitely generated abelian groups, we know that $A = (\mathbb{Z}/k\mathbb{Z}) \oplus (\mathbb{Z}/l\mathbb{Z})$ or $A = \mathbb{Z}/k\mathbb{Z}$. For the second case, we have the S.E.S.

$$0 \to \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^{m+n}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0.$$

For the first case, by the injection we know one of k, l must be greater or equal than p^m. W.l.o.g., we assume it k. By the same argument as the first part of this problem, the S.E.S. is

$$0 \to \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{1 \mapsto (p^t,1)} (\mathbb{Z}/p^{m+t}\mathbb{Z}) \oplus (\mathbb{Z}/p^{n-t}\mathbb{Z}) \xrightarrow{(1,0) \mapsto 1} \mathbb{Z}/p^n\mathbb{Z} \to 0$$

where $0 \le t < n$.

In the third case, we know that map $\mathbb{Z} \to A$ is injective hence there is one free element in A. But $A/\mathbb{Z}$ has rank 0 hence A has rank exactly 1. By the fundamental theorem of f.g. abelian groups, either we have torsion elements or not. If there is no torsion element, then we have

$$0 \to \mathbb{Z} \xrightarrow{k \mapsto nk} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

If there is a torsion element in A, suppose it is of order m, then the map $\mathbb{Z} \to A$ must maps 1 to $(1, a)$, where a is an element in the torsion part. Otherwise suppose $1 \mapsto (d, a)$ for some integer d, then $A/\mathbb{Z} = \mathbb{Z}/d\mathbb{Z} \oplus T$ where T is the torsion of A. But $\mathbb{Z}/d\mathbb{Z} \oplus T \ncong \mathbb{Z}/n\mathbb{Z}$. Thus the S.E.S. is

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

□

Exercise 2.1.9 (Exercise 2.1.16)**.**

1. Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
2. Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Solution. (i) Suppose A meets each path-component of X, then for each path-component X_i of X, we have some nonempty subspace $A_i = A \cap X_i$. For any point $x \in X_i$, we have some point $y \in A_i$ s.t. there is a path $\alpha : I \to X_i$ connecting x and y. Thus $x - y = \partial(\alpha) \in \operatorname{Im} \partial$ and hence $C_0(X)/\operatorname{Im} \partial = C_0(A)/\operatorname{Im} \partial$, which means $H_0(X, A) = 0$.

Conversely, we have the following S.E.S.

$$0 \to C.(A) \to C.(A) \to C.(X, A) \to 0,$$

which induces the following exact sequence

$$H_0(A) \to H_0(X) \to H_0(X, A) \to 0.$$

Since $H_0(X, A) = 0$, we know that the map $H_0(A) \to H_0(X)$ is surjective. Thus for any $\sigma \in H_0(X)$, $\sigma \in H_0(A)$, which is actually for any singular map $\sigma : \{*\} \to X$, there is a $\tau \in C_0(A)$ s.t. $\sigma - \tau \in \operatorname{Im}(\partial)$. Suppose $\partial(\alpha) = \sigma - \tau$, and suppose $\tau = \sum_{i=1}^n \epsilon_i \tau_i$ where each $\tau_i : \{*\} \to A$ and $\epsilon = \pm 1$, we have some points $x_i \in A$ defined by $\tau_i(*) = x_i$. Then α is the path connecting x to some point x_j in A. Hence meets each path-component of X.

(ii) Similar to above, we have the following exact sequence

$$H_1(A) \to H_1(X) \to H_1(X,A) \to H_0(A) \to H_0(X),$$

thus $H_1(X,A)$ iff $H_1(A) \to H_1(X)$ is surjective and $H_0(A) \to H_0(X)$ is injective. By prop 2.6, we know for each path-connected component $A_i \subseteq X_i$, $H_0(A_i) \to H_0(X_i) = \mathbb{Z}$ is injective, hence each path-component of X contains at most one path-component of A. □

Exercise 2.1.10 (Exercise 2.1.17).

1. Compute the homology groups $H_n(X,A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
2. Compute the groups $H_n(X,A)$ and $H_n(X,B)$ for X a closed orientable surface of genus two with A and B the circles shown.

Solution. (i) Suppose $A = \{x_1, \cdots, x_m\}$, then

$$H_n(A) = \begin{cases} \mathbb{Z}^m & \text{when } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Also we know that

$$H_n(S^2) = \begin{cases} \mathbb{Z} & \text{when } n = 0,2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & \text{when } n = 0,2 \\ \mathbb{Z}^2 & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By the long exact sequences of relative homology groups, we have

$$0 \to \mathbb{Z} \to H_2(S^2, A) \to 0 \to 0 \to H_1(S^2, A) \to \mathbb{Z}^m \to \mathbb{Z} \to H_0(S^2, A) \to 0$$

and

$$0 \to \mathbb{Z} \to H_2(S^1 \times S^1, A) \to 0 \to \mathbb{Z}^2 \to H_1(S^1 \times S^1, A) \to \mathbb{Z}^m \to \mathbb{Z} \to H_0(S^1 \times S^1, A) \to 0.$$

By previous problem, we know

$$H_0(S^2, A) = H_0(S^1 \times S^1, A) = 0$$

since S^2 and $S^1 \times S^1$ are path-connected. Therefore by the exactness $H_2(S^2, A) = \mathbb{Z}$, $H_1(S^2, A) = \mathbb{Z}^{m-1}$, and $H_2(S^1 \times S^1, A) = \mathbb{Z}$. To be more precise, the map

(ii) Notice that (X,A) and (X,B) are good pairs, hence $H_n(X,A) \cong H_n(X/A)$ and $H_n(X,B) \cong H_n(X/B)$. But $X/A \cong T^2 \vee T^2 \cong T^2 \coprod T^2/\{x,y\}$, hence $H_n(X,A) = H_n(T^2, \{x\}) \oplus H_n(T^2, \{y\})$. By previous part,

$$H_n(T^2, \{x\}) = \begin{cases} \mathbb{Z} & \text{when } n = 2 \\ \mathbb{Z}^2 & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$H_n(X,A) = \begin{cases} \mathbb{Z}^2 & \text{when } n = 2 \\ \mathbb{Z}^4 & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $X/B \cong T^2/\{x,y\}$, hence $H_n(X,A) = H_n(T^2, \{x\}) \oplus H_n(T^2, \{x,y\})$, therefore

$$H_n(X,B) = \begin{cases} \mathbb{Z} & \text{when } n = 2 \\ \mathbb{Z}^3 & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Exercise 2.1.11 (Exercise 2.1.20)**.** Show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Solution. Since $X \cong X \times \{1\}$ is a good subspace of CX, hence $H_n(CX, X) \cong H_n(CX/X)$. But $CX/X \cong SX$, hence we have the following exact sequence

$$\tilde{H}_n(X) \to \tilde{H}_n(CX) \to \tilde{H}_n(CX, X) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(CX).$$

But CX is contractible, so $\tilde{H}_n(CX) = 0$, therefore $\tilde{H}_n(CX, X) \to \tilde{H}_{n-1}(X)$ is an isomorphism. Thus

$$\tilde{H}_{n-1}(X) \cong \tilde{H}_n(CX, X) \cong \tilde{H}_n(CX/X) \cong \tilde{H}_n(SX).$$

Furthermore, we know that $CX \cup CX$ by identifying their bases is SX, hence we have the case $k = 2$. When we consider $k + 1$ cones CX with their bases identified, we have

$$\tilde{H}_{n+1}(\bigcup_{i=1}^{k+1} CX) = \tilde{H}_{n+1}(\bigcup_{i=1}^{k} CX \cup CX) = \tilde{H}_{n+1}(\bigcup_{i=1}^{k} CX/X).$$

But $\bigcup_{i=1}^{k} CX/X = \bigvee_{i=1}^{k} SX$, hence

$$\tilde{H}_{n+1}(\bigcup_{i=1}^{k+1} CX) = \tilde{H}_{n+1}(\vee_{i=1}^{k} SX) = \bigoplus_{i=1}^{k} \tilde{H}_{n+1}(SX) \cong \bigoplus_{i=1}^{k} \tilde{H}_n(X).$$

□

Exercise 2.1.12 (Exercise 2.1.21)**.** Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\tilde{H}_n(X) \to \tilde{H}_{n+1}(SX)$.

Solution. For any $\sigma \in C_n(X)$, we know that σ is a continuous map $\Delta^n \to X$. Thus we can have a natural $C(\sigma) : \Delta^{n+1} \to CX$, where Δ^{n+1} can be viewed as the set $C(\Delta^n) = \{(x, t) \in \Delta^n \times I \mid |x| + |t| \leq 1\}$, $CX = X \times I/(X \times \{1\})$ and $C(\sigma)(x, t) = (\sigma(x), 0)$. Therefore we define a map (by taking the linear span)

$$C : C_n(X, \{*\}) \to C_{n+1}(CX, X).$$

The map is actually an embedding of a singular simplex, hence it is induced by the injection $\iota : X \to CX$ hence C is a chain map. We also have a map

$$q_* : C_{n+1}(CX, X) \to C_{n+1}(SX, \{*\}),$$

induced by the quotient map $q : (CX, X) \to (SX, \{*\})$. Hence the composition $q_* \circ C$ induces a map $\tilde{H}_n(X) = H_n(X, \{*\}) \to \tilde{H}_{n+1}(SX) = H_{n+1}(SX, \{*\})$. It suffices to prove that the map is the isomorphism.

For the triple $(\{*\}, X, CX)$, we have

$$0 \to C_\bullet(X, \{*\}) \to C_\bullet(CX, \{*\}) \to C_\bullet(CX, X) \to 0,$$

hence an exact sequence

$$H_n(X, \{*\}) \to H_n(CX, \{*\}) \to H_n(CX, X) \xrightarrow{\delta} H_{n-1}(X, \{*\}).$$

By the definition of δ, for any $[\sigma] \in H_{n-1}(X, \{*\})$, $\delta \circ C([\sigma]) = [\sigma]$ since for any $\tau \in C_n(CX, \{*\})$ s.t. $q_*(\tau) = C(\sigma)$, $\delta \circ C([\sigma]) = [\partial(\tau)] = [\sigma]$. q_* induces an isomorphism because the excision theorem (prop 2.22), hence the composition is an isomorphism. □

Exercise 2.1.13 (Exercise 2.1.22)**.** Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n spheres:

1. If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
2. $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n - 1$ or $n + 1$.

3. If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Solution. The defining S.E.S. of relative homology

$$0 \to C_\bullet(X^{n-1}) \to C_\bullet(X^n) \to C_\bullet(X^n, X^{n-1}) \to 0$$

induces a long exact sequence

$$\cdots \to H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n, X^{n-1}) \to H_{i-1}(X^{n-1}) \to \cdots.$$

Since X^{n-1} is a subcomplex,

$$H_i(X^n, X^{n-1}) \cong H_i(X^n/X^{n-1}) \cong H_i(\bigvee_{j\in J} S^n) \cong \bigoplus_{j\in J} H_i(S^n).$$

1. If $\dim X = n$, then $X = X^n$. When $n = 1$, foe each connected component Y of X, it is a wedge sum of circles, hence
$$H_i(Y, Y^0) \cong \tilde{H}_i(\bigvee_{j\in J} S^1) = \begin{cases} \bigoplus_{j\in J}\mathbb{Z} & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$
Therefore for the case $n = 1$, $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free. Suppose it is true for $n-1$, then we consider the case n. The long exact sequence gives
$$\cdots \to H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) = \bigoplus_{i\in I}\mathbb{Z} \to H_{n-1}(X^{n-1}) \to \cdots$$
where $H_n(X^{n-1}) = 0$ by induction hypothesis. This implies that $H_n(X^n)$ is isomorphic to a subgroup of $\bigoplus_{i\in I}\mathbb{Z}$, which means $H_n(X^n)$ is a free abelian group. The long exact sequence for $i > n$ gives $H_i(X) = 0$ for $i > n$ immediately by induction hypothesis.
2. We first prove that $H_n(X^n) \cong H_n(X)$ if there are no $n+1$-cells. It is obviously true for $\dim X \le n+1$. For $m > n$, the long exact sequence
$$\cdots \to H_{n+1}(X^m, X^{m-1}) \to H_n(X^{m-1}) \to H_n(X^m) \to H_n(X^m, X^{m-1}) \to \cdots$$
satisfies that $H_{n+1}(X^m, X^{m-1}) = H_n(X^m, X^{m-1}) = 0$, so one has
$$H_n(X^n) \cong \cdots \cong H_n(X^{m-1}) \cong H_n(X^m) \cong H_n(X^{m+1}) \cong \cdots.$$
However, since X is of finite dimension, this sequence is stationary at some point, which is $H_n(X)$. Where there are no cells of dimension $n-1$, the long exact sequence gives
$$\cdots \to H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) = \bigoplus_{i\in I}\mathbb{Z} \to H_{n-1}(X^{n-1}) \to \cdots$$
where $H_n(X^{n-1}) = 0$ by previous part and $H_{n-1}(X^{n-1}) = 0$ because X has no $n-1$-cells so that $X^{n-1} = X^{n-2}$. The abelian group $H_n(X^n, X^{n-1})$ is free with basis in bijective correspondence with the n-cells, so is $H_n(X^n)$ by the exactness.
3. For $m > n+1$, the long exact sequence
$$\cdots \to H_{n+1}(X^m, X^{m-1}) \to H_n(X^{m-1}) \to H_n(X^m) \to H_n(X^m, X^{m-1}) \to \cdots$$
satisfies that $H_{n+1}(X^m, X^{m-1}) = H_n(X^m, X^{m-1}) = 0$, so one has
$$H_n(X^{n+1}) \cong \cdots \cong H_n(X^{m-1}) \cong H_n(X^m) \cong H_n(X^{m+1}) \cong \cdots \cong H_n(X).$$
Again, the long exact sequence
$$\cdots \to H_{n+1}(X^{n+1}, X^n) \to H_n(X^n) \to H_n(X^{n+1}) \to H_n(X^{n+1}, X^n) = 0 \to \cdots.$$
This means $H_n(X^{n+1}) = H_n(X)$ is a surjective image of $H_n(X^n)$, which is $\mathbb{Z}^k$ by the first part.

□

Exercise 2.1.14 (Exercise 2.1.23)**.** Show that the second barycentric subdivision of a Δ complex is a simplicial complex. Namely, show that the first barycentric subdivision produces a Δ complexwith the property that each simplex has all its vertices distinct, then show that for a Δ complex with this property, barycentric subdivision produces a simplicial complex.

Solution. □

Exercise 2.1.15 (Exercise 2.1.27)**.** Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.

1. Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
2. For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ s.t. fg and gf are homotopic to the identity through maps of pairs.

Solution. (i) Since f are homotopy equivalence, we have isomorphisms $f_* : H_n(X) \to H_n(Y)$ and $f_* : H_n(A) \to H_n(B)$. Furthermore, by the naturality we have $f_* : H_n(X, A) \to H_n(Y, B)$ s.t. the following commutative diagram

$$\begin{array}{ccccccccc} H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\delta} & H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y,B) & \xrightarrow{\delta} & H_{n-1}(B) & \xrightarrow{i_*} & H_{n-1}(Y) \end{array}$$

with both rows exact. Hence by five lemma, the middle induced homomorphism is an isomorphism.

(ii) We first prove the following: If $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences, then $f : (X, \bar{A}) \to (Y, \bar{B})$ is also a homotopy equivalence. For any $a \in \bar{A}$, then for any open set U s.t. $f(a) \in U$, $f^{-1}(U)$ is open hence $f^{-1}(U) \cap A \neq \emptyset$, thus $U \cap f(A) \subseteq U \cap B \neq \emptyset$. Therefore $f(\bar{A}) \subseteq \bar{B}$. By assumptions, we have $g : Y \to X$ and $H : X \times I \to X$ s.t. $H|_{X\times\{0\}} = g \circ f$ and $H|_{X\times\{1\}} = \mathrm{id}$, and $H(A, I) \subseteq A$. But also by previous discussion, $H(\bar{A}, I) \subseteq \bar{A}$, hence H is the homotopy we want.

Back to the problem, suppose we have a homotopy $\underline{f : (D^n, S^{n-1})} \hookrightarrow (D^n, D^n - \{0\})$, then this can be a homotopy $f : (D^n, \overline{S^{n-1}}) = (D^n, S^{n-1}) \hookrightarrow (D^n, \overline{D^n - \{0\}}) = (D^n, D^n)$. Thus by part (a), f induces a isomorphism between the relative homology groups of (D^n, S^{n-1}) and (D^n, D^n). The last one is trivial, but $H_{n+1}(D^n, S^{n-1}) = H_{n+1}(S^{n+1}) = \mathbb{Z}$, a contradiction. □

Remark. The second part is a counterexample to Prop 2.19., saying that the map described in the second part is not a homotopy between maps of pairs.

Exercise 2.1.16 (Exercise 2.1.29)**.** Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution. We know that

$$H_n(S^2) = \begin{cases} \mathbb{Z} & \text{when } n = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & \text{when } n = 0, 2 \\ \mathbb{Z}^2 & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{H}_n(S^1 \vee S^1 \vee S^2) = \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2)$, we know

$$H_n(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & \text{when } n = 0, 2 \\ \mathbb{Z}^2 & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions.

The covering space of $S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$ is obviously $\mathbb{R}^2 \to S^1 \times S^1$, $(x, y) \mapsto (x - [x], y - [y])$ where $[x]$ is the greatest integer smaller or equal than x. For $S^1 \vee S^1 \vee S^2$, we choose the base point as the connecting point. Notice that the covering space of S^2 is still S^2, therefore the covering space of $S^1 \vee S^1 \vee S^2$ is $S^2 \cup_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} (\mathbb{R} \cup_{\{0\}} \mathbb{R})$, where the space is first gluing two lines at their origins, then gluing each integer with a sphere on the lines. Thus, since $\mathbb{R}^2$ is contractible, the only nontrivial homology group of $\mathbb{R}^2$ is $H_0(\mathbb{R}^2) = \mathbb{Z}$, while the 2nd homology group of covering space of $S^1 \vee S^1 \vee S^2$ is nontrivial since it is the wedge product of an S^2 with some other space. Hence their universal covering spaces do not have the same homology groups. □

Exercise 2.1.17 (Exercise 2.1.31)**.** Using the notation of the five-lemma, give an example where the maps α, β, δ, and ϵ are zero but γ is nonzero. This can be done with short exact sequences in which all the groups are either $\mathbb{Z}$ or 0.

Solution. The diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

satisfies the conditions. □

2.2 Computations and Applications

Exercise 2.2.1. Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and the southern hemispheres of S^n to the southern hemisphere via f.

Solution. By Exercise 0.0.6, any map $f : D^n \to D^n$ is contractible, hence the map

$$f \cup_{S^{n-1}} f : S^n \to S^n$$

defined as stated in the problem is also contractible. Therefore $\deg f \cup_{S^{n-1}} f = 0$. If f does not admit a fixed point, neither does $f \cup_{S^{n-1}} f$. Therefore

$$(-1)^{n+1} = \deg f \cup_{S^{n-1}} f = 0,$$

a contradiction. □

Exercise 2.2.2 (Exercise 2.2.2)**.** Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{RP}^{2n}$ has a fixed point. Construct maps $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Solution. Suppose that f does not admit any fixed point, then $\deg f = (-1)^{2n+1} = -1$. Similarly, if f does not admit any point s.t. $f(x) = -x$, then $-f$ does not admit fixed point, and $\deg -f = (-1)^{2n+1} = -1$. But $\deg -f = -\deg f$, hence if there is no point s.t. $f(x) = x$ or $f(x) = -x$, then $-1 = \deg f = 1$, a contradiction.

We denote $\mathbb{RP}^{2n} \cong S^{2n}/\sim$, where $x \sim y \in S^{2n}$ if and only if $x = -y$. Thus, S^{2n} is a covering space of $\mathbb{RP}^{2n}$, and for any map $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$, we have a map $\tilde{f} : S^{2n} \to S^{2n}$ s.t. the following diagram commutes

$$\begin{tikzcd}
S^{2n} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & S^{2n} \arrow[d, "\pi"] \\
\mathbb{RP}^{2n} \arrow[r, "f"] & \mathbb{RP}^{2n}
\end{tikzcd}$$

where π is the quotient map. This is by the lifting property of the covering space. By previous discussion, there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$, hence the point is a fixed point after quotient.

Suppose $\mathbb{R}^{2n}$ has the standard basis $\{e_1, \cdots, e_{2n}\}$, and suppose linear map $\mathcal{A}$ has matrix

$$A := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_n \end{pmatrix}$$

over the given basis where $R_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$ for some $0 \neq \theta_i$'s. Thus $\mathcal{A}$ is an isometry preserving the origin, i.e. maps S^{2n-1} to S^{2n-1}, hence it induces a map $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$. To see it has no fixed point, it suffices to prove that the matrix does not have eigenvector except 0. Suppose that $a_1e_1 + \cdots + a_{2n}e_{2n} \neq 0$ is an eigenvector of $\mathcal{A}$, then by the definition $a_{2i-1}e_{2i-1} + a_{2i}e_{2i}$ is an eigenvector of R_i. However, R_i has only eigenvector 0, hence $\mathcal{A}$ has only eigenvector 0. □

Exercise 2.2.3 (Exercise 2.2.3)**.** Let $f : S^n \to S^n$ be a map of degree 0. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radically inward.

Solution. If there is no point x s.t. $f(x) = x$, then $\deg f = (-1)^{n+1}$. Similarly, if there is no point y s.t. $f(y) = -y$, then $-\deg f = \deg -f = (-1)^{n+1}$. Both cases contradict the fact that $\deg f = 0$.

Define a new continuous vector field

$$G(x) := \frac{F(x)}{\|F(x)\|}$$

since $F(x) \neq 0$ for all x. If there is no point on ∂D^n where F points radically inward, then there is no points $x, y \in S^{n-1}$ with $G|_{S^{n-1}}(x) = x$ and $G|_{S^{n-1}}(y) = -y$, where $G|_{S^{n-1}} = G \circ \iota$, and $\iota : S^{n-1} \hookrightarrow D^n$ is the canonical embedding. Hence $(G|_{S^{n-1}})_*$ is not of degree 0. But the funtoriality tells us that $(G|_{S^{n-1}})_* = G_* \circ \iota_* = 0$, a contradiction. □

Exercise 2.2.4.

Solution. Space-filling curve. □

Exercise 2.2.5 (2.2.5)**.** Show that any two reflections of S^n across different n-dimensional hyperplanes are homotopic, in fact homotopic through reflections.

Solution. We view S^n as a subspace

$$S^n := \{(a_1, \cdots, a_{n+1}) \mid \sum_{i=1}^{n+1} a_i^2 = 1\} \subseteq \mathbb{R}^{n+1},$$

then it suffices to show that every reflection is homotopic (through reflections) to the reflection

$$\begin{aligned} r_{n+1} : \mathbb{R}^{n+1} &\to \mathbb{R}^{n+1} \\ (a_1, \cdots, a_{n+1}) &\mapsto (a_1, \cdots, -a_{n+1}). \end{aligned}$$

Consider any reflection

$$r : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},$$

by definition it fixes and is determined totally by some hyperplane

$$H_{(c_1,\cdots,c_{n+1})} := \{(x_1, \cdots, x_{n+1}) \mid (c_1, \cdots, c_{n+1}) \cdot (x_1, \cdots, x_{n+1}) = 0\},$$

W.L.O.G., one can take $(c_1, \cdots, c_{n+1}) \in S^n$. Thus by the path-connectivity of S^n, there is a path

$$\gamma : I \to S^n$$

with $\gamma(0) = (c_1, \cdots, c_{n+1})$ and $\gamma(1) = (0, \cdots, 0, 1)$. Take a homotopy

$$H_\gamma : S^n \times I \to S^n$$

where $H_\gamma(-, t)$ is the reflection determined by the hyperplane $H_{\gamma(t)}$. To be precise, for any

$$\boldsymbol{x} = (\boldsymbol{x} \cdot \gamma(t))\gamma(t) + (\boldsymbol{x} - (\boldsymbol{x} \cdot \gamma(t))\gamma(t)) \in \mathbb{R}^{n+1}$$

one defines

$$H_\gamma(\boldsymbol{x}, t) := -(\boldsymbol{x} \cdot \gamma(t))\gamma(t) + (\boldsymbol{x} - (\boldsymbol{x} \cdot \gamma(t))\gamma(t)).$$

□

Remark. This is more clear with a point view of homotopy theory: $\pi_n(S^n) = \mathbb{Z}$ and a homotopy equivalence class of maps is determined completely by its degree.

Exercise 2.2.6 (2.2.7)**.** For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$, show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R} - \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R} - \{0\}) \cong \mathbb{Z}$ is id or $-$id according to whether the determinant of f is positive or negative.

Solution. We try to show that two types of elementary row operations are homotopic to id relative to $\mathbb{R} - \{0\}$.

Scaling For a constant $c > 0$, there is a homotopy

$$H_m(t, -) := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & (c-1)t+1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Adding For a constant $c > 0$, there is a homotopy

$$H_a(t, -) := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & ct & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Thus, for any invertible matrix A, there is a factorisation

$$A = E_k E_{k-1} \cdots E_1 \epsilon$$

where each E_i is an elementary matrix either taking a multiplication or taking an addition, both for positive constant c, and ϵ is some finite multiplication of reflections r_i along $\boldsymbol{e}_i$, of which the finite number depends on the determinant. By previous discussion, $A \simeq \epsilon$.

Then it suffices to show that ϵ induces $\pm$id. The isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\})$ comes from the long exact sequence

$$\cdots \to \tilde{H}_n(\mathbb{R}^n) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \to \tilde{H}_{n-1}(\mathbb{R}^n) \to \cdots$$

where the isomorphism sends a relative class $[\sigma]$ to $[\partial\sigma]$. Take

$$\begin{aligned} \sigma : \Delta^n &\to \mathbb{R}^n \\ e_0 &\mapsto (-1, \cdots, -1) \\ e_i &\mapsto (-1, \cdots, -1, 2, -1, \cdots, -1) \end{aligned}$$

to be a representative of a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$, W.L.O.G., we say this is $[\sigma] = 1 \in \mathbb{Z}$ and $[\partial\sigma] = 1 \in \mathbb{Z} \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\})$. Thus, for any reflection $r : \mathbb{R}^n \to \mathbb{R}^n$, $r|_{S^n} : S^n \to S^n$ is also a reflection. The interpretation of the induced map is

$$\begin{aligned} r_* : H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) &\to \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \\ [\sigma] &\mapsto [\partial(r(\sigma))], \end{aligned}$$

which is also

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \xrightarrow{(r|_{S^n})_*} \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}).$$

One knows that $(r|_{S^n})_*$ is of degree 1, and done. □

Exercise 2.2.7 (Exercise 2.2.8)**.** A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Solution. It is easy to construct a homotopy between $a_n z^n$ and z^n

$$\begin{aligned} H_0 : S^2 \times I &\to S^2 \\ (z, t) &\mapsto a_n^t z^n \end{aligned}$$

if $a_n \neq 0$, hence by Example 2.32 and Prop 2.33, we know $f_n(z) = a_n z^n$ is of degree n. Then for any polynomial $f(z) = a_n z^n + \cdots + a_1 z + a_0$ we define

$$\begin{aligned} H_1 : S^2 \times I &\to S^2 \\ (z, t) &\mapsto a_n z^n + t(f(z) - a_n z^n), \end{aligned}$$

which satisfies $H_1|_{S^2\times\{0\}} = a_n z^n$ and $H_1|_{S^2\times\{1\}} = f(z) - a_n z^n$. If H_1 is continuous then it is a homotopy, hence we know $f(z)$ is homotopic to z^n, whose degree is n. And then we are done.

Since polynomials are continuous over $\mathbb{C}$, we know that H_1 is continuous over $\mathbb{C} \times I$. Therefore it suffices to prove $f(z)$ is continuous at each point in $\{\infty\} \times I$. Since $a_n \neq 0$, $H_1(\infty, t) = \infty$ for all $t \in I$. Thus for any fixed $t_0 \in I$

$$\begin{aligned} \lim_{z\to\infty} H_1(z, t_0) &= \lim_{z\to\infty} a_n z^n + t_0(f(z) - a_n z^n) \\ &= \lim_{z\to\infty} z^n \left(a_n + \frac{t_0(f(z) - a_n z^n)}{z^n} \right) \\ &= \lim_{z\to\infty} z^n \lim_{z\to\infty} a_n + \frac{t_0(f(z) - a_n z^n)}{z^n} \\ &= a_n \lim_{z\to\infty} z^n = \infty, \end{aligned}$$

i.e. H_1 is continuous and it is a homotopy.

Then we suppose $f(z) = a_n(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$ be the linear facorization, then we can find some sufficiently small open neighborhood $U_i \cong \mathbb{C}$ of z_i s.t. $f(z)|_{U_i} = (z - z_i)^{m_i} g_i(z)$ where $g_i(z)$ is holomorphic on U_i and has no zero on U_i. Actually, if we consider the Taylor expansion of $f(z)$ near z_i then it is straight forward. Let $h(z) = \sqrt[m_i]{g_i(z)}$ and denote $w = (z - z_i)h(z)$, then $f(z) = w^{m_i}$. Thus we have the following diagram

$$\begin{tikzcd} U_i \arrow[r, "f|_{U_i}"] \arrow[d, "z\mapsto w"'] & S^{2n} \\ U \arrow[ur, "w\mapsto w^{m_i}"'] & \end{tikzcd}$$

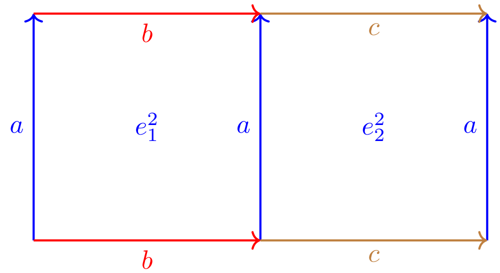
where $z \mapsto w$ is holomorphic hence homeomorphic and U is an open neighborhood of the origin. Thus, the local degree of f at z_i is the same as the degree of $w \mapsto w^{m_i}$, which is m_i, hence we are done. □

Exercise 2.2.8 (Exercise 2.2.9b)**.** Compute the homology groups of the following 2-complexes:

1. The quotient of S^2 obtained by identifying north and south poles to a point.
2. $S^1 \times (S^1 \vee S^1)$.
3. The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.
4. The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Solution. 1.

2. We impose the cell structure where we have $X_0 := \{P\}$, $X_1 := \{a, b, c\}$ and $X_2 := \{e_1^2, e_2^2\}$.



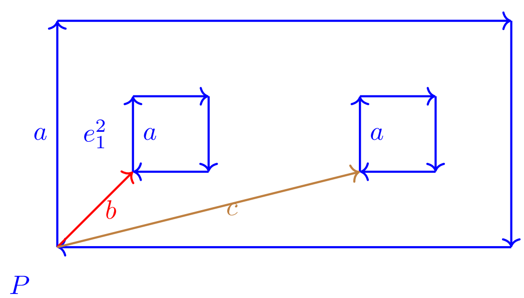
Thus we have the chain complex

$$0 \to H_2(X^2, X^1) = \mathbb{Z}^2 \xrightarrow{d_2} H_1(X^1, X^0) = \mathbb{Z}^3 \xrightarrow{d_1} H_0(X^0) = \mathbb{Z} \to 0.$$

By the cellular boundary, $d_2(e_1^2) = a + b - a - b = 0$ and $d_2(e_2^2) = a + c - a - c = 0$, and $d_1(a) = d_1(b) = d_1(c) = P - P = 0$. Thus, by the definition

$$H_i(S^1 \times (S^1 \vee S^1)) = \begin{cases} \mathbb{Z}^2 & i = 2, \\ \mathbb{Z}^3 & i = 1, \\ \mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. We impose the cell structure where we have $X_0 := \{P\}$, $X_1 := \{a, b, c\}$ and $X_2 := \{e_1^2\}$.



Thus one has the chain complex

$$0 \to H_2(X^2, X^1) = \mathbb{Z} \xrightarrow{d_2} H_1(X^1, X^0) = \mathbb{Z}^3 \xrightarrow{d_1} H_0(X^0) = \mathbb{Z} \to 0.$$

By the cellular boundary, $d_2(e_1) = a + c - a - c + b - a - b = -a$ and $d_1(a) = d_1(b) = d_1(c) = P - P = 0$. Therefore

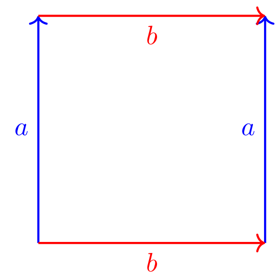
$$H_i(X) = \begin{cases} \mathbb{Z} & i = 2, \\ \mathbb{Z}^2 & i = 1, \\ \mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4.

$\square$

Exercise 2.2.9 (Exercise 2.2.12)**.** Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering space that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

Solution. We first impose CW complexes onto $S^1 \times S^1$ and S^2. Let $S^1 \times S^1$ be the CW complex $X_0 := \{P\}$, $X_1 := \{a, b\}$ and $X_2 := \{e^2\}$



and let S^2 be the CW complex $Y_0 := \{Q\}$, $Y_2 := \{f^2\}$. Thus the quotient map π sends a, b, P to Q, and sends e^2 homeomorphically to f^2, thus the π induces a homomorphism mapping the generator of $H_2(X^2, X^1)$ to the generator of $H_2(Y^2, Y^1)$ and hence $H_2(X^2, X^1) \to H_2(Y^1, Y^0)$ is surjective. Since S^2 does not admit any 1-cell, $H_2(Y^1, Y^0) = 0$ and therefore $H_2(Y^1, Y^0) = H_2(S^2) = \mathbb{Z}$. If it is not injective, then the kernel is $m\mathbb{Z}$ for some nonzero integer m, hence by the first isomorphism theorem, $\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$, a contradiction. Hence $H_2(S^1 \times S^1) \to H_2(S^2)$ is an isomorphism.

On the other hand, suppose that $f : S^2 \to S^1 \times S^1$ is a map, since the covering space of $S^1 \times S^1$ is $\mathbb{R}^2$, hence we have the following factorization

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "\pi"] \\ S^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \times S^1. \end{tikzcd}$$

Then it suffices to prove that $\tilde{f}$ is nullhomotopic. Nevertheless by Exercise 0.10, $\tilde{f}$ is automatically nullhomotopic. □

Exercise 2.2.10 (Exercise 2.2.14)**.** A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be 0 when n is even. When n is odd, show there exist even maps of any given even degree.

Solution. First suppose that x_i is in the preimage of x, then we can find a small neighborhood of x_i, denoted by U_i. Let $-U_i := \{-x \mid x \in U_i\}$, then $-U_i$ is a small neighborhood of $-x_i$ which is homeomorphic to U_i. But f is even, so $f|_{U_i} = f|_{-U_i}$, hence by local degree formula, $\deg f = \sum_i \deg f|_{x_i} + \sum_i \deg f|_{-x_i} = 2\sum_i \deg f|_{x_i}$ is an even number.

Here since f is even, we have a factorization

$$\begin{tikzcd} \mathbb{RP}^n \arrow[dr, "\tilde{f}"] & \\ S^n \arrow[u, "\pi"] \arrow[r, "f"] & S^n \end{tikzcd}$$

where π is the canonical quotient $S^n \to \mathbb{RP}^n$. By the funcoriality, we have

$$H_n(S^n) \cong \mathbb{Z} \xrightarrow{\pi_*} H_n(\mathbb{RP}^n) \xrightarrow{\tilde{f}_*} H_n(S^n),$$

hence when n is even, $H_n(\mathbb{RP}^n) = 0$ hence f is of degree 0.

Then if n is odd, $H_n(\mathbb{RP}^n) \cong \mathbb{Z}$ and $\pi_*(1) = 2$ because S^2 is a two sheeted covering space. Thus it suffices to construct a map $\tilde{f} : \mathbb{RP}^n \to S^n$ s.t. $\tilde{f}_*(1) = k$ for any given k. But the map $q : \mathbb{RP}^n \to S^n$ by quotient $\mathbb{RP}^{n-1}$ has the property that $q_*(1) = 1$, hence for any given k, we can construct

$$\mathbb{RP}^n \xrightarrow{i} \bigvee_k \mathbb{RP}^n \xrightarrow{q_k} S^n$$

where i is the injection of $\mathbb{RP}^n$ into the wedge product, and q_k is the k copy of q. Therefore the composition $q_k \circ i$ is of "degree" k, and we are done. $\square$

Exercise 2.2.11 (2.2.19)**.** Compute $H_i(\mathbb{RP}^n/\mathbb{RP}^m)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{RP}^n$ with $\mathbb{RP}^m$ as its m skeleton.

Solution. Give $\mathbb{RP}^n$ a CW structure by attach one i-cell to $\mathbb{RP}^{i-1}$ by winding its boundary two rounds. Thus the complex associated to this CW structure is

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \cdots \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0$$

if n is even, and is

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

if n is odd.

1. For m being even, $m+1$ is odd, so the complex is

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots \leftarrow 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \cdots \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0$$

if n is even, and is

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots \leftarrow 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

if n is odd. In this case,

$$H_i(\mathbb{RP}^n/\mathbb{RP}^m) = \begin{cases} H_i(\mathbb{RP}^n) & i = 0, i > m, \\ 0 & \text{otherwise.} \end{cases}$$

2. For m being odd, $m+1$ is even, so the complex is

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots \leftarrow 0 \xleftarrow{2=0} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \cdots \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0$$

if n is even, and is

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots \leftarrow 0 \xleftarrow{2=0} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

if n is odd. In this case,

$$H_i(\mathbb{RP}^n/\mathbb{RP}^m) = \begin{cases} H_i(\mathbb{RP}^n) & i = 0, i > m, \\ \mathbb{Z} & i = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Exercise 2.2.12 (Exercise 2.2.20)**.** For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Solution. By theorem A.6, we have a natural CW structure on $X \times Y$, where all the n-cells are of the form $e^i_\alpha \times f^j_\beta$ where e^i_α and f^j_β are i-cell and j-cell of X and Y respectively, and $i + j = n$. Thus

$$\begin{aligned}
\chi(X \times Y) &= \sum_{k=0}^{m+n} (-1)^k \#\{k\text{-cells in } X \times Y\} \\
&= \sum_{k=0}^{m+n} \sum_{i+j=k} (-1)^k (\#\{i\text{-cells in } X\}) \cdot (\#\{j\text{-cells in } Y\}) \\
&= \left(\sum_{k=0}^{m} (-1)^i \#\{i\text{-cells in } X\} \right) \cdot \left(\sum_{k=0}^{n} (-1)^j \#\{j\text{-cells in } Y\} \right) \\
&= \chi(X)\chi(Y),
\end{aligned}$$

where m, n are the dimension of X and Y respectively. $\square$

Exercise 2.2.13 (Exercise 2.2.21)**.** If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Solution. Give X the CW as following: e^n_α is an n-cell of both A and B if and only if e^n_α is an n-cell of $A \cap B$, i.e. A and B have compatible CW structure. Since X is finite, then there is a sufficiently large N s.t.

$$X = \bigcup_{n=0}^{N} X^n = \bigcup_{n=0}^{N} A^n \cup B^n,$$

hence

$$\chi(X) = \sum_{n=0}^{N} \chi(A^n \cup B^n).$$

By the given CW structure, if we count the number of n-cells in $A^n \cup B^n$, we can first count those in A then those in B. But this makes the n-cells in $A^n \cap B^n$ to be counted twice, hence

$$\#\{n\text{-cells in } A \cup B\} = \#\{n\text{-cells in } A\} + \#\{n\text{-cells in } B\} - \#\{n\text{-cells in } A \cap B\}.$$

Therefore

$$\begin{aligned}
\chi(X) &= \sum_{n=0}^{N} \chi(A^n \cup B^n) \\
&= \sum_{n=0}^{N} (-1)^n \#\{n\text{-cells in } A \cup B\} \\
&= \sum_{n=0}^{N} (-1)^n (\#\{n\text{-cells in } A\} + \#\{n\text{-cells in } B\} - \#\{n\text{-cells in } A \cap B\}) \\
&= \sum_{n=0}^{N} (-1)^n \#\{n\text{-cells in } A\} + \sum_{n=0}^{N} (-1)^n \#\{n\text{-cells in } B\} - \sum_{n=0}^{N} (-1)^n \#\{n\text{-cells in } A \cap B\} \\
&= \chi(A) + \chi(B) - \chi(A \cap B).
\end{aligned}$$

□

Exercise 2.2.14 (Exercise 2.2.22)**.** For a finite CW complex X and $p : \tilde{X} \to X$ an n-sheeted covering space, show that $\chi(\tilde{X}) = n\chi(X)$.

Solution. It suffices to know the CW structure of $\tilde{X}$. Since we have the characteristic map $\varphi_\alpha : e^n_\alpha \to X$ and $\pi_1(e^n_\alpha, \{*\}) = 0$, there is a lift

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
e^n_\alpha \arrow[r, "\varphi_\alpha"] \arrow[ur, dashed] & X,
\end{tikzcd}$$

therefore each $p^{-1}(e^n_\alpha)$ is homeomorphic to e^n_α. And since $p : \tilde{X} \to X$ an n-sheeted covering space, $p|_{p^{-1}(e^n_\alpha)}$ is also an n-sheeted covering space of e^n_α. Thus we have the induced CW complex on $\tilde{X}$. By the definition,

$$\begin{aligned}
\chi(\tilde{X}) &= \sum_{k=0}^{N} (-1)^k \#\{k\text{-cells in } \tilde{X}\} \\
&= \sum_{k=0}^{N} (-1)^k n \#\{k\text{-cells in } X\} \\
&= n \sum_{k=0}^{N} (-1)^k \#\{k\text{-cells in } X\} \\
&= n\chi(X).
\end{aligned}$$

□

Exercise 2.2.15 (2.2.24)**.** Suppose we build S^2 from a finite collection of polygons by identifying edges in pairs. Show that in the resulting CW structure on S^2 the 1 skeleton cannot be either of the two graphs shown, with five and six vertices. [This is one step in a proof that neither of these graphs embeds in $\mathbb{R}^2$.]

Solution. For all homologies in this problem, we consider the group in coefficient $\mathbb{Q}$.

For the graph K_5, suppose on the contrary it is the 1-skeleton of a CW structure on S^2, then by

$$v - e + f = \chi(S^2) = 2,$$

putting $v = 5$ and $e = 10$ one has $f = 7$, which means there are exactly 7 2-cells. Since $H_1(S^2) = 0$, each edge, in the image of the boundary of the 2-cells, has to be counted even times. If one edge is counted more than twice, then along this edge the space is not locally flat, which is not the sphere. Therefore each edge is counted at most twice. However this means

$$\text{sum of all edges in each 2-cell} \leq 20,$$

where the left hand side is at least $3 \times 7 = 21$, a contradiction.

For the graph $K_{3,3}$, the argument is very similar. □

Exercise 2.2.16 (Exercise 2.2.29)**.** The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X onto two copies of R. Also compute the relative groups $H_i(R, M_g)$.

Solution. Let A and B be two copies of R, then $A \cap B = M_g$. The homology groups of M_g are given by the textbook, and the homology groups of R are

$$H_n(R) = \begin{cases} \mathbb{Z} & \text{when } n = 0 \\ \mathbb{Z}^g & \text{when } n = 1 \\ 0 & \text{otherwise,} \end{cases}$$

which can be proved by induction on g and Mayer-Vietoris sequence (or more easily R is homotopic to $\bigvee_{i=1}^g S^1$, whose homology groups can be easily compute). Thus by Mayer-Vietoris sequence of reduced homology groups we have the following long exact sequence

$$0 \to \tilde{H}_3(X) \to 0 \to \mathbb{Z} \to \tilde{H}_2(X) \to \mathbb{Z}^{2g} \to \mathbb{Z}^g \oplus \mathbb{Z}^g \to \tilde{H}_1(X) \to 0 \to 0 \to \tilde{H}_0(X) \to 0.$$

Consider two inclusions $i : A \to X$, $j : B \to X$, if we denote the generators of $H_1(M_g)$ be $a_1, b_1, \cdots, a_g, b_g$, where a's for the "meridian" and b's for the "parallel", then $i^*([a_k]) = j^*([a_k]) = 0$ and $i^*([b_k]) = j^*([b_k]) = [b_k]$. Thus the map $\mathbb{Z}^{2g} \to \mathbb{Z}^g \oplus \mathbb{Z}^g$ is an isomorphism, therefore $\tilde{H}_2(X) = \mathbb{Z}$ and $\tilde{H}_1(X) = 0$.

We then take the relative groups $H_i(-, M_g)$ of the Mayer-Vietoris sequence

$$H_3(R, M_g) \to H_3(X, M_g) \to 0 \to H_2(R, M_g) \to H_2(X, M_g) \to 0 \to H_1(R, M_g) \to H_1(X, M_g) \to 0.$$

But M_g is a good subspace and $X/M_g \cong R/M_g \vee R/M_g$, hence $H_1(X, M_g) = H_1(R, M_g) \oplus H_1(R, M_g)$. But by the long exact sequence, $H_i(R, M_g) \cong H_i(X, M_g)$ for all i, hence $H_i(R, M_g)$ has only nontrivial homology group $H_0(R, M_g) = \mathbb{Z}$. □

Exercise 2.2.17 (Exercise 2.2.32)**.** For SX the suspension of X, show by a Mayer-Vietoris sequence that there are isomorphism $\tilde{H}_n(SX) = \tilde{H}_{n-1}(X)$ for all n.

Solution. Let $A = CX$ and $B = CX$ be two subspace of SX where $A \cap B \cong X \times [0, 1] \simeq X$. And let x_0 is a point in $A \cap B$. Thus the Mayer-Vietoris sequence tells us that

$$H_n(A \cap B, \{x_0\}) \to H_n(A, \{x_0\}) \oplus H_n(B, \{x_0\}) \to H_n(SX, \{x_0\}) \to H_{n-1}(A \cap B, \{x_0\}).$$

Since A and B are contractible, $H_n(A, \{x_0\}) \oplus H_n(B, \{x_0\}) = 0$, hence

$$H_n(A \cap B, \{x_0\}) \to 0 \to H_n(SX, \{x_0\}) \to H_{n-1}(A \cap B, \{x_0\}) \to 0$$

is exact, which means that $\tilde{H}_n(SX) = H_n(SX, \{x_0\}) = H_{n-1}(A \cap B, \{x_0\}) = H_{n-1}(X, \{x_0\}) = \tilde{H}_{n-1}(X)$. □

Exercise 2.2.18 (Exercise 2.2.40)**.** From the long exact sequence of homology groups associated to the short exact sequence of chain complexes $0 \to C_i(X) \xrightarrow{n} C_i(X) \to C_i(X;\mathbb{Z}/n\mathbb{Z}) \to 0$ deduce immediately that there are short exact sequences

$$0 \to H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z}) \to n-\mathrm{Torsion}(H_{i-1}(X)) \to 0$$

where $n-\mathrm{Torsion}(H_{i-1}(X))$ is the kernel of the map $G \xrightarrow{n} G, g \mapsto ng$. Use this to show that $\tilde{H}_i(X;\mathbb{Z}/p\mathbb{Z}) = 0$ for all i and all primes p iff $\tilde{H}_i(X)$ is a vector space over $\mathbb{Q}$ for all i.

Solution. By snake lemma, we have the long exact sequence

$$H_i(X) \xrightarrow{n} H_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z}) \to H_{i-1}(X),$$

hence we have the injective homomorphism $0 \to H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z})$ because of the exactness at $H_i(X) \xrightarrow{n} H_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z})$. Then we consider the abelian group A in the S.E.S.

$$0 \to H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z}) \to A \to 0$$

is just $H_i(X;\mathbb{Z}/n\mathbb{Z})$ modulo the image $H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z})$, which is also $H_i(X;\mathbb{Z}/n\mathbb{Z})$ modulo the image $H_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z})$. By the exactness at

$$H_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z}) \to H_{i-1}(X),$$

the image of $H_i(X) \to H_i(X;\mathbb{Z}/n\mathbb{Z})$ is the kernel of $H_i(X;\mathbb{Z}/n\mathbb{Z}) \to H_{i-1}(X)$, hence by the first isomorphism theorem, A is isomorphic to the image of $H_i(X;\mathbb{Z}/n\mathbb{Z}) \to H_{i-1}(X)$, which is exactly $H_{i-1}(X) \xrightarrow{n} H_{i-1}(X)$.

If $\tilde{H}_i(X;\mathbb{Z}/p\mathbb{Z}) = 0$ for all i and all primes p, we know by previous argument $H_i(X)$ does not have any p-torsion. Thus, each p is invertible as coefficient of $H_i(X)$, which means $H_i(X)$ is a $\mathbb{Q}$ module. Conversely, if $H_i(X)$ is a $\mathbb{Q}$ module, obviously $C_i(X) \xrightarrow{p} C_i(X)$ is an isomorphism for all p, hence so is $H_i(X) \xrightarrow{n} H_i(X)$, therefore in the S.E.S. above, $H_i(X;\mathbb{Z}/p\mathbb{Z}) = 0$ since $n-\mathrm{Torsion}(H_{i-1}(X))$ is 0. □

Exercise 2.2.19 (2.2.41)**.** For X a finite CW complex and F a field, show that the Euler characteristic $\chi(X)$ can also be computed by the formula $\chi(X) = \sum_n (-1)^n \dim H_n(X;F)$, the alternating sum of the dimensions of the vector spaces $H_n(X;F)$.

Solution. Consider the complex of F-coefficient associated with the CW structure

$$0 \leftarrow C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \leftarrow 0$$

with $C_i := H_i(X^i, X^{i-1}; F)$. Since X is a finite CW complex, each C_i is a finite dimensional vector space. Construct basis of each vector space satisfying

1. $\{b_{i,1}, \cdots, b_{i,l_i}, z_{i,1}, \cdots, z_{i,m_i}, c_{i,1}, \cdots, c_{i,n_i}\}$ is a basis of C_i, such that $\{b_{i,1}, \cdots, b_{i,l_i}\}$ is a basis of B_i, extending to a basis $\{b_{i,1}, \cdots, b_{i,l_i}, z_{i,1}, \cdots, z_{i,m_i}\}$ of Z_i, then extending to a basis of C_i.
2. ∂_{i+1} sends $c_{i+1,1}, \cdots, c_{i+1,n_{i+1}}$ to $b_{i,1}, \cdots, b_{i,l_i}$, and hence $n_{i+1} = l_i$.

By definition, the number of i-cells is equal to the dimension of C_i, hence

$$\begin{aligned}
\chi(X) &= \sum_{i=0}^{n} (-1)^i \dim C_i \\
&= \sum_{i=0}^{n} (-1)^i (l_i + m_i + n_i) \\
&= \sum_{i=1}^{n} (-1)^i (l_i + m_i + l_{i-1}) + l_0 + m_0 \\
&= \sum_{i=0}^{n} (-1)^i m_i \\
&= \sum_{i=0}^{n} (-1)^i \dim H_i^{\mathrm{CW}}(X;F) \\
&= \sum_{i=0}^{n} (-1)^i \dim H_i(X;F).
\end{aligned}$$

□

Appendix 2.A Homology and Fundamental Group

Appendix 2.B

Exercise 2.B.1 (2.B.10)**.** Use the transfer sequence for the covering $S^\infty \to \mathbb{RP}^\infty$ to compute $H_n(\mathbb{RP}^\infty; \mathbb{Z}_2)$.

Solution. We consider the 2-sheeted cover $p : S^\infty \to \mathbb{RP}^\infty$, which gives us a transfer sequence

$$0 \to C_n(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau} C_n(S^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_\#} C_n(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to 0,$$

inducing a long exact sequence

$$\cdots \to H_n(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau_*} H_n(S^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_n(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to \cdots.$$

Since

$$H_n(S^\infty; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{when } n = 0 \\ 0 & \text{otherwise} \end{cases},$$

we have

$$H_n(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = H_{n-1}(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \cdots = H_0(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = H_0(S^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

□

Exercise 2.B.2 (2.B.11)**.** Use the transfer sequence for the covering $X \times S^\infty \to X \times \mathbb{RP}^\infty$ to produce isomorphism $H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}_2) \cong \bigoplus_{i \le n} H_n(X; \mathbb{Z}_2)$.

Solution. We consider the two-sheeted covering

$$\begin{aligned} \mathrm{id} \times p : X \times S^\infty &\to X \times \mathbb{RP}^\infty \\ (x, t) &\mapsto (x, p(t)) \end{aligned}$$

where p is the two-sheeted covering $p : S^\infty \to \mathbb{RP}^\infty$. Thus we have the transfer sequence

$$0 \to C_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau} C_n(X \times S^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mathrm{id}_\# \times p_\#} C_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to 0,$$

which induces a long exact sequence

$$\cdots \to H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau_*} H_n(X \times S^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to \cdots.$$

Since S^∞ does not admit an n-cell, we know that

$$H_n(X \times S^\infty; \mathbb{Z}/2\mathbb{Z}) \cong H_n(X; \mathbb{Z}/2\mathbb{Z})$$

by cell structure. But the transfer map $C_n(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to C_n(S^\infty; \mathbb{Z}/2\mathbb{Z})$ is zero, hence

$$\tau_* : H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_n(X \times S^\infty; \mathbb{Z}/2\mathbb{Z})$$

is also 0. Therefore we have a S.E.S.

$$0 \to H_n(X \times S^\infty; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \to 0.$$

However, all these groups are of coefficient $\mathbb{Z}/2\mathbb{Z}$, hence they are vector spaces over $\mathbb{Z}/2\mathbb{Z}$. So the S.E.S. splits, and we have that

$$\begin{aligned} H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) &= H_n(X \times S^\infty; \mathbb{Z}/2\mathbb{Z}) \oplus H_{n-1}(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \\ &= H_n(X; \mathbb{Z}/2\mathbb{Z}) \oplus H_{n-1}(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}), \end{aligned}$$

implying that $H_n(X \times \mathbb{RP}^\infty; \mathbb{Z}_2) \cong \bigoplus_{i \le n} H_n(X; \mathbb{Z}_2)$ by induction. □

Appendix 2.C Simplicial Approximation

Exercise 2.C.1 (2.C.2)**.** Use the Lefschetz fixed point theorem to show that a map $S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal mao $x \mapsto -x$.

Solution. Suppose there is a map $f : S^n \to S^n$, then the induced map

$$f_* : H_*(S^n) \to H_*(S^n)$$

consists of $f_0 : H_0(S^n) \to H_0(S^n)$ and $f_n : H_n(S^n) \to H_n(S^n)$, and all others are 0. Since f is continuous and S^n is connected, f_0 is id. Hence

$$\tau(f) = 1 + (-1)^n \deg f.$$

By Lefschetz, provided $\deg f \neq (-1)^{n+1}$, which is the degree of the antipodal mao $x \mapsto -x$, $\tau(f) \neq 0$ and f admits a fixed point. □

Exercise 2.C.2 (Exercise 2.C.4)**.** If X is a finite simplicial complex and $f : X \to X$ is a simplicial homeomorphism, show that the Lefschetz number $\tau(f)$ equals the Euler characteristic of the set of fixed points of f. In particular, $\tau(f)$ is the number of fixed points if the fixed points are isolated.

Solution. We first barycentrically subdivide X to make the fixed point set a subcomplex, denoted by X^m, and we also denote the fixed simplex by X^f. Since X is a finite simplicial complex, so is X^m, i.e. $C_n(X^m)$ is a finitely generated free abelian group for all n. Thus each $(f_*)_n : C_n(X^m) \to C_n(X^m)$ is a permutation matrix since f is simplicial and hence it permutes all the n-dimensional simplex. Therefore

$$\mathrm{Tr}\ (f_*)_n : C_n(X^m) \to C_n(X^m) = \#\{\text{fiexed } n\text{-dimensional simplex}\} = \dim C_n(X^f).$$

Hence we proved that

$$\sum_n (-1)^n \mathrm{Tr}\ (f_*)_n : C_n(X^m) \to C_n(X^m) = \sum_n (-1)^n \dim C_n(X^f),$$

where the right hand side is the Euler characteristic of the set of fixed points of f. So it suffices to prove that

$$\sum_n (-1)^n \mathrm{Tr}\ (f_*)_n : C_n(X^m) \to C_n(X^m) = \sum_n (-1)^n \mathrm{Tr}\ (\overline{f_*})_n : H_n(X^m) \to H_n(X^m).$$

This can be proved similarly to theorem 2.44, simply because Tr is additive. For each C_n, we have S.E.S.'s $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, hence

$$\mathrm{Tr}\ (f_*)_n = \mathrm{Tr}\ (f_*)_n|_{Z_n} + \mathrm{Tr}\ (f_*)_n|_{B_{n-1}}$$

and

$$\mathrm{Tr}\ (f_*)_n|_{Z_n} = \mathrm{Tr}\ (\overline{f_*})_n + \mathrm{Tr}\ (f_*)_n|_{B_{n-1}}.$$

After the substitution and taking the alternating sum, we have

$$\sum_n (-1)^n \mathrm{Tr}\ (f_*)_n = \sum_n (-1)^n \mathrm{Tr}\ (\overline{f_*})_n.$$

Hence it suffices to prove that the barycentric subdivision X^m exists. Suppose d is the largest dimension of simplices intersecting with the fixed points, and take any point p in the interior of the simplex which is fiexed and in the boundary of the fixed points. Denote one of these simplices by Δ. We take this p as the barycenter, and for all the subsimplices of Δ, if it intersects with fixed points, take a point in the boundary of the intersection as the barycenter, otherwise take an arbitrary point. Do this finitely many times then the largest dimension of simplices intersecting with the fixed points reduces to $d - 1$. Hence by induction the subdivision exists. □

Exercise 2.C.3 (Exercise 2.C.5)**.** Let M be a closed orientable surface embedded in $\mathbb{R}^3$ in such a way that reflection across a plane P defines a homeomorphism $r : M \to M$ fixing $M \cap P$, a collection of circles. Is it to homotope r to have no fixed points?

Solution. We first compute the Lefschetz number. Just by previous problem, we know that the Lefschetz number is the Euler characteristic of the set of fixed points. But for each circle, the Euler characteristic is 0, hence the disjoint union of finitely many circles. Thus $\tau(r) = 0$, which means r could be homotopic to some map without fixed point.

Denote the set of fixed points by $N = \coprod_i S^1$. Since M be a closed orientable surface, we have an open neighborhood U of N, where $U = \coprod_i S^1 \times (-1, 1)$. We shall homotope the map r only on U, rotating each circle a little so that there is no fixed point admitted. Define $H : M \times [0, 1] \to M$

$$H(x,t) := \begin{cases} x & \text{if } x \notin U \\ (y + \pi t|s|, s) & \text{if } x = (y, s) \in \coprod_i S^1 \times (-1, 1) \end{cases}$$

where we denote $S^1 \times (-1, 1) = \{(y, s) \mid y \in [0, 2\pi), s \in (-1, 1)\}$. Thus $H|_{M\times\{0\}} = r$, and $H|_{M\times\{1\}}$ has no fixed point, because $H|_{M\times\{1\}}$ apperantly has no fixed point on $M - N$ since all points have to be mapped to another side. Also $H|_{M\times\{1\}}$ has no fixed point on N since the homotopy rotating all the circles. □

Exercise 2.C.4. Let M_g be the orientable surface of genus g and let N_{2g} be the non-orientable surface of genus $2g$. Prove that for any continuous map $f : N_{2g} \to M_g$, the induced map

$$f_* : H_2(N_{2g}; \mathbb{Z}/2\mathbb{Z}) \to H_2(M_g; \mathbb{Z}/2\mathbb{Z})$$

is trivial.

Solution. □

Chapter 3

Cohomology

3.1 Cohomology Groups

Exercise 3.1.1. Show that $\mathrm{Ext}(H, G)$ is a contravariant functor of H for fixed G, and a covariant functor of G for fixed H.

Solution. Given any group homomorphism $f : H_1 \to H_2$. Suppose $P_\bullet \to H_1$ and $F_\bullet \to H_2$ are free resolutions respectively. Then by Lemma 3.1, $f : H_1 \to H_2$ induces a map of chains

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_2"] & P_0 \arrow[r] \arrow[d, "f_0"] & H_1 \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & H_2 \arrow[r] & 0.
\end{tikzcd}$$

Applying functor $\mathrm{Hom}(-, G)$ to the diagram, one has

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(H_2, G) \arrow[r] \arrow[d, "f^*"] & \mathrm{Hom}(F_0, G) \arrow[r] \arrow[d, "f_0^*"] & \mathrm{Hom}(F_1, G) \arrow[r] \arrow[d, "f_1^*"] & \mathrm{Hom}(F_2, G) \arrow[r] \arrow[d, "f_2^*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(H_1, G) \arrow[r] & \mathrm{Hom}(P_0, G) \arrow[r] & \mathrm{Hom}(P_1, G) \arrow[r] & \mathrm{Hom}(P_2, G) \arrow[r] & \cdots.
\end{tikzcd}$$

Since $f^* = \mathrm{Hom}(f_\bullet, G)$ is a cochain map, it induces maps between cohomology groups

$$f^* : H^*(\mathrm{Hom}(F_\bullet, G)) \to H^*(\mathrm{Hom}(P_\bullet, G)).$$

However, by definition $H^1(\mathrm{Hom}(P_\bullet, G)) \cong \mathrm{Ext}(H_1, G)$ and $H^1(\mathrm{Hom}(F_\bullet, G)) \cong \mathrm{Ext}(H_2, G)$, hence one has the induced map

$$\mathrm{Ext}(f, G) : \mathrm{Ext}(H_2, G) \to \mathrm{Ext}(H_1, G).$$

This map is independent of the choice of resolutions, because by Lemma 3.1, different construction of $f_\bullet : P_\bullet \to F_\bullet$ differs only by one chain homotopy, which induces 0 map between cohomology groups.

Then one needs to verify that given $H_1 \xrightarrow{f} H_2 \xrightarrow{g} H_3$ of abelian group homomorphisms,

$$\mathrm{Ext}(g \circ f, G) = \mathrm{Ext}(f, G) \circ \mathrm{Ext}(g, G).$$

However, $\mathrm{Ext}(g \circ f, G)$ is the construction from a resolution $P_\bullet \to H_1$ to $Q_\bullet \to H_3$, and it could come from a resolution $P_\bullet \to H_1$ to $F_\bullet \to H_2$, then to $Q_\bullet \to H_3$. Since taking cohomology groups is a functor, this gives exactly

$$\mathrm{Ext}(g \circ f, G) = \mathrm{Ext}(f, G) \circ \mathrm{Ext}(g, G).$$

On the other hand, given any group homomorphism $f : G_1 \to G_2$, by the fact that $\mathrm{Hom}(K, -)$ is a covariant functor, for any free resolution $P_\bullet \to H$, one has

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(H, G_1) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}(P_0, G_1) \arrow[r] \arrow[d, "f_0"] & \mathrm{Hom}(P_1, G_1) \arrow[r] \arrow[d, "f_1"] & \mathrm{Hom}(P_2, G_1) \arrow[r] \arrow[d, "f_2"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(H, G_2) \arrow[r] & \mathrm{Hom}(P_0, G_2) \arrow[r] & \mathrm{Hom}(P_1, G_2) \arrow[r] & \mathrm{Hom}(P_2, G_2) \arrow[r] & \cdots
\end{tikzcd}$$

as a map of cochain complex. The map on the first cohomology group is $\mathrm{Ext}(H, f)$. In short, it is the composition of two functors $H^1 \circ \mathrm{Hom}(P_\bullet, -)$ applied on f. And the fact that

$$\mathrm{Ext}(H, g \circ f) = \mathrm{Ext}(H, g) \circ \mathrm{Ext}(H, f)$$

comes from that both H^1 and $\mathrm{Hom}(P_\bullet, -)$ are functors. □

Exercise 3.1.2. Show that the maps $n : G \to G$ and $n : H \to H$ multiplying each element by the integer n induce multiplication by n in $\mathrm{Ext}(H, G)$.

Solution. Take the free resolution $F_\bullet$ as

$$0 \to F_1 \to F_0 \to H \to 0,$$

where $F_i = 0$ for all $i \neq -1, 0, 1$. Hence by Lemma 3.1 we have the following map chain complexes

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r] \arrow[d, "n"] & F_0 \arrow[r] \arrow[d, "n"] & H \arrow[r] \arrow[d, "n"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & H \arrow[r] & 0. \end{tikzcd}$$

Take $\mathrm{Hom}(-, G)$ we have

$$\begin{tikzcd} F_1^* \arrow[d, "n"] & F_0^* \arrow[l, "i^*"'] \arrow[d, "n"] & H^* \arrow[l] \arrow[d, "n"] & 0 \arrow[l] \\ F_1^* & F_0^* \arrow[l, "i^*"'] & H^* \arrow[l] & 0. \arrow[l] \end{tikzcd}$$

Since $\mathrm{Ext}(H, G) = F_1^*/\mathrm{Im}\ i^*$, by the commutativity of the diagram $n(\mathrm{Im}\ i^*) \subseteq \mathrm{Im}\ i^*$, thus we have that $n : F_1^* \to F_1^*$ induces a homomorphism $n : \mathrm{Ext}(H, G) \to \mathrm{Ext}(H, G)$.

On the other hand, for any group K, the map $n : G \to G$ induces $n : \mathrm{Hom}(K, G) \to \mathrm{Hom}(K, G)$ by $f \mapsto f \circ n$. Thus for the projective resolution

$$0 \to F_1 \to F_0 \to H \to 0,$$

we also have

$$\begin{tikzcd} F_1^* \arrow[d, "n"] & F_0^* \arrow[l, "i^*"'] \arrow[d, "n"] & H^* \arrow[l] \arrow[d, "n"] & 0 \arrow[l] \\ F_1^* & F_0^* \arrow[l, "i^*"'] & H^* \arrow[l] & 0, \arrow[l] \end{tikzcd}$$

hence it also induces a homomorphism $n : \mathrm{Ext}(H, G) \to \mathrm{Ext}(H, G)$. □

Exercise 3.1.3 (Exercise 3.1.3)**.** Regarding $\mathbb{Z}/2\mathbb{Z}$ as a module over the ring $\mathbb{Z}/4\mathbb{Z}$, constructing a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/4\mathbb{Z}$ and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is nonzero for all n.

Solution. We first have a $\mathbb{Z}/4\mathbb{Z}$-module homomorphism $\pi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. We have a unique, canonical inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/\mathbb{Z}$ and π is defined to be the quotient map $\pi : \mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/4\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $\pi : 1, 3 \mapsto 1$ and $0, 2 \mapsto 0$. Then we have a chain complex

$$\cdots \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0,$$

where in each term $\mathbb{Z}/4\mathbb{Z}$, the image of π is $\mathbb{Z}/2\mathbb{Z}$ and it happens to be the kernel of π. Hence this is a free resolution. Take the functor $\mathrm{Hom}(-, \mathbb{Z}/2\mathbb{Z})$, and by noticing that $\mathrm{Hom}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, we have a cochain complex

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\bar{\pi}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\bar{\pi}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\bar{\pi}} \cdots,$$

where $\bar{\pi} : \mathrm{Hom}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $f \mapsto f \circ \pi$ is actually the zero map. Hence $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for all n. □

Exercise 3.1.4 (Exercise 3.1.5)**.** Regarding a cochain $\varphi \in C^1(X;G)$ as a function from paths in X to G, show that if φ is a cocycle, then

(i) $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$,

(ii) φ takes the value 0 on constant paths,

(iii) $\varphi(f) = \varphi(g)$ if $f \simeq g$,

(iv) φ is a coboundary if iff $\varphi(f)$ depends only on the endpoints of f for all f.

Solution. (i) Let $\sigma : [v_0, v_1, v_2] \to X$ be the 2-simplex s.t. $f : [v_0, v_1] \to X$ and $g : [v_1, v_2] \to X$, thus $f \cdot g : [v_0, v_2] \to X$. Then

$$\partial\sigma = f - f \cdot g + g.$$

Hence since φ is a cocycle, we have

$$0 = \delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(f - f \cdot g + g) = \varphi(f) - \varphi(f \cdot g) + \varphi(g).$$

(ii) Suppose f is constant. Then $f \cdot g = g$ for any $g \in C_1(X)$. By previous part

$$\varphi(f) = \varphi(f \cdot g) - \varphi(g) = \varphi(g) - \varphi(g) = 0.$$

(iii) Suppose $f \simeq g$, then $f \cdot g^{-1}$ bounds some compact region. We define the simplex $\sigma : [v_0, v_1, v_2] \to X$ s.t. $\sigma|_{[v_0,v_1]} = f$, $\sigma|_{[v_1,v_2]} = f(1) = g(1)$ and $\sigma|_{[v_0,v_2]} = g$. Thus by previous parts

$$\varphi(f) - \varphi(g) = \varphi(f) - \varphi(g) + \varphi(f(1)) = \varphi(\partial\sigma) = \delta\varphi(\sigma) = 0,$$

implying $\varphi(f) = \varphi(g)$.

(iv) Suppose there is some $\psi \in C^0(X;G)$ with $\phi = \delta\psi$, then $\phi = \psi \circ \partial : C_1(X) \to G$. Thus for any $f \in C_1(X)$,

$$\phi(f) = \psi \circ \partial(f) = \psi(f(0) - f(1)) = \psi(f(0)) - \psi(f(1)).$$

Conversely, if $\varphi(f)$ depends only on the endpoints of f. Then for each path-connected component X_i of X, take a base point $x_i \in X_i$, and then define

$$\psi(x) := \varphi(f)$$

where f is a path connecting x_i and x, i.e. $f(0) = x_i$ and $f(1) = x$, and we linearly extend it to be an element in $C^0(X;G)$. ψ is well-defined because φ depends only on end points. Suppose $x = f(1) = g(0)$, then by part (i),

$$\begin{aligned}
\psi(f(1)) - \psi(g(0)) &= \psi(f(0)) - \psi(g(1)) + \psi(f(1)) - \psi(g(0)) - \psi(f(0)) + \psi(g(1)) \\
&= (\psi(f(0)) - \psi(g(1))) - (\psi(g(0)) - \psi(g(1))) + (\psi(f(0)) - \psi(f(1))) \\
&= \varphi(\partial f \cdot g) - \varphi(\partial f) - \varphi(\partial g) \\
&= 0.
\end{aligned}$$

Then one can verify immediately that $\varphi = \delta\psi$. □

Exercise 3.1.5 (Exercise 3.1.8)**.** Many basic homology arguments work just as well for cohomology even though maps go in the opposite direction. Verify this in the following cases:

1. Compute $H^i(S^n;G)$ by induction on n in two ways: using the long exact sequence of a pair, and using the Mayer-Vietoris sequence.

2. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms $H^n(X,A;G) \cong \tilde{H}^n(X/A;G)$ for all n.

3. Show that if A is a retract of X then $H^n(X;G) \cong H^n(A;G) \oplus H^n(X,A;G)$.

Solution. (i) Consider the pair (D^n, S^{n-1}), then we have S.E.S.

$$0 \to C^\bullet(D^n, S^{n-1}; G) \to C^\bullet(D^n; G) \to C^\bullet(S^{n-1}; G) \to 0$$

then a long exact sequence

$$\to H^i(D^n, S^{n-1}; G) \to H^i(D^n; G) \to H^i(S^{n-1}; G) \to H^{i+1}(D^n, S^{n-1}; G) \to \cdots .$$

Since D^n is contractible, $H^i(D^n; G) = 0$, hence $H^{i+1}(D^n, S^{n-1}; G) \cong H^i(S^{n-1}; G)$. But (D^n, S^{n-1}) is a good pair, therefore $H^{i+1}(D^n, S^{n-1}; G) \cong H^{i+1}(S^n; G)$, which is what we want.

On the other hand, let A, B be two subspace of S^n where A is S^n minus the north pole and B is S^n minus the south pole. Induced by the S.E.S.

$$0 \to C^\bullet(A + B; G) \to C^\bullet(A; G) \oplus C^\bullet(B; G) \to C^\bullet(A \cap B; G) \to 0$$

there is a long exact sequence

$$\to H^{i-1}(A \cap B; G) \to H^i(A + B; G) \to H^i(A; G) \oplus H^i(B; G) \to H^i(A \cap B; G) \to \cdots$$

where $A \cap B \simeq S^{n-1}$ and A and B are contratible. Hence $H^{i-1}(S^{n-1}; G) \cong H^i(S^n; G)$, which is what we want

$$H_n(S^n) = \begin{cases} \mathbb{Z} & \text{when } n = 0, n \\ 0 & \text{otherwise} \end{cases}$$

.

(ii) We have a canonical quotient map $q : X \to X/A$ inducing a map between pairs $q : (X, A) \to (X/A, A/A)$. By the naturality, we have that

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X, A); G) \arrow[r] & H^n(X, A; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(X, A), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X/A, A/A); G) \arrow[r] \arrow[u, "(q_*)^*"] & H^n(X/A, A/A; G) \arrow[r, "h"] \arrow[u, "q^*"] & \mathrm{Hom}(H_n(X/A, A/A), G) \arrow[r] \arrow[u, "(q_*)^*"] & 0.
\end{tikzcd}$$

Since A is a closed subspace of X that is a deformation retract of some neighborhood, q induces an isomorphism $H_n(X, A) \cong H_n(X/A, A/A)$, hence the two $(q_*)^*$ in the diagram are isomorphism, so by five lemma q^* is also an isomorphism. But we also have an isomorphism $H^n(X/A, A/A; G) \cong \tilde{H}^n(X/A; G)$, therefore we are done.

(iii)By the functority of homology groups, the retraction r induces a splitting long exact sequence

$$\to H_{i-1}(X, A) \to H_i(A) \to H_i(X) \to H_i(X, A) \to \cdots ,$$

where $i_* : H_i(A) \to H_i(X)$ is an injection, which means the long exact sequence should be

$$\to H_{i-1}(X, A) \to 0 \to H_i(A) \to H_i(X) \to H_i(X, A) \to 0 \to \cdots .$$

Thus $H_i(X) = H_i(A) \oplus H_i(X, A)$. Thus by the naturality we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X); G) \arrow[r] \arrow[d] & H^n(X; G) \arrow[r, "h"] \arrow[d] & \mathrm{Hom}(H_n(X), G) \arrow[d] \arrow[r] & {} \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X, A); G) \oplus \mathrm{Ext}(H_{n-1}(A); G) \arrow[r] & H^n(X, A; G) \oplus H^n(A; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(X, A), G) \oplus \mathrm{Hom}(H &
\end{tikzcd}$$

We know that $\mathrm{Ext}(H_{n-1}(X); G) \to \mathrm{Ext}(H_{n-1}(X, A); G) \oplus \mathrm{Ext}(H_{n-1}(A); G)$ and $\mathrm{Hom}(H_n(X), G) \to \mathrm{Hom}(H_n(X, A), G) \oplus \mathrm{Hom}(H_n(A), G)$ are two isomorphisms, hence by the five lemma, the middle map is an isomorphism. □

Exercise 3.1.6 (Exercise 3.1.9)**.** Show that if $f : S^n \to S^n$ has degree d then $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.

Solution. By the naturality, we have that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}(\tilde{H}_{n-1}(S^n);G) & \longrightarrow & \tilde{H}^n(S^n;G) & \xrightarrow{h} & \operatorname{Hom}(\tilde{H}_n(S^n),G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f_*)^*} & & \\ 0 & \longrightarrow & \operatorname{Ext}(\tilde{H}_{n-1}(S^n);G) & \longrightarrow & \tilde{H}^n(S^n;G) & \xrightarrow{h} & \operatorname{Hom}(\tilde{H}_n(S^n),G) & \longrightarrow & 0. \end{array}$$

Notice that $\tilde{H}_{n-1}(S^n) = 0$, hence $\operatorname{Ext}(\tilde{H}_{n-1}(S^n);G) = 0$, therefore the map h is an isomorphism. For each $g \in \operatorname{Hom}(\tilde{H}_n(S^n),G)$, we have that $(f_*)^*(g) = g \circ f_*$. But $f : S^n \to S^n$ has degree d implies that for any $k \in \tilde{H}_n(S^n)$, $(f_*)^*(g)(k) = (g \circ f_*)(k) = g(dk) = dg(k)$, so $(f_*)^*(g) = dg$. Therefore by the commutativity of the diagram, for any $\alpha \in \tilde{H}^n(S^n;G)$

$$\begin{aligned} f^*(\alpha) &= h^{-1} \circ (f_*)^* \circ h(\alpha) \\ &= h^{-1}(d \cdot h(\alpha)) \\ &= d \cdot h^{-1}(h(\alpha)) = d\alpha. \end{aligned}$$

Or one can use $H_*(S^n)$ instead of $\tilde{H}_*(S^n)$, only when $n = 1$ there is a difference. However by the fact $\operatorname{Ext}(\mathbb{Z},G) = 0$, the argument still works. □

Exercise 3.1.7 (3.1.11)**.** Let X be a Moore space $M(\mathbb{Z}_m, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m.

1. Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\tilde{H}_i(-,\mathbb{Z})$ for all i, but not on $H^{n+1}(-,\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.

2. Show that the inclusion $S^n \to X$ induces the trivial map on $\tilde{H}^i(-,\mathbb{Z})$, but not on $H_n(-,\mathbb{Z})$.

Solution. (i) By the cell structure, we know that

$$\tilde{H}_i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & \text{when } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\tilde{H}_i(S^{n+1},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{when } i = n+1, \\ 0 & \text{otherwise.} \end{cases},$$

hence every induced map $f_* : \tilde{H}_*(X,\mathbb{Z}) \to \tilde{H}_*(S^{n+1},\mathbb{Z})$ must be trivial. Then by the universal coefficient theorem we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}(H_n(X);\mathbb{Z}) & \longrightarrow & H^{n+1}(X;\mathbb{Z}) & \xrightarrow{h} & \operatorname{Hom}(H_{n+1}(X),\mathbb{Z}) & \longrightarrow & 0 \\ & & {\scriptstyle (f_*)^*}\uparrow & & {\scriptstyle f^*}\uparrow & & {\scriptstyle (f_*)^*}\uparrow & & \\ 0 & \longrightarrow & \operatorname{Ext}(H_n(S^{n+1});\mathbb{Z}) & \longrightarrow & H^{n+1}(S^{n+1};\mathbb{Z}) & \xrightarrow{h} & \operatorname{Hom}(H_{n+1}(S^{n+1}),\mathbb{Z}) & \longrightarrow & 0, \end{array}$$

which is actually

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \xrightarrow{h} & 0 & \longrightarrow & 0 \\ & & {\scriptstyle (f_*)^*}\uparrow & & {\scriptstyle f^*}\uparrow & & {\scriptstyle (f_*)^*}\uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{h} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Since that f maps the $n+1$-cell homeomorphically to the $n+1$-cell, hence f^* is surjective, hence f does not induce a trivial map on $H^{n+1}(-,\mathbb{Z})$.

From the second commutative diagram above we have that the left and the right maps are trivial, if the split is natural, then f^* also need to be trivial. A contradiction.

(ii) First the inclusion $g : S^n \to X$ send the n-cell of S^n to the n-cell of X, where both of the n-cells are generators of $H_n(S^n,\mathbb{Z})$ and $H_n(X,\mathbb{Z})$. But both of the groups are not trivial, hence g_* is not trivial.

The only nontrivial cohomology group of S^n is $\tilde{H}^n(S^n,\mathbb{Z})$, but $\tilde{H}^i(X,\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. There is no nontrivial homomorphism $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}$, hence g^* are all trivial. □

Exercise 3.1.8 (3.1.13)**.** Let $\langle X, Y\rangle$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$. Using proposition 1.B.9, show that if X is a connected CW complex and G is an abelian group, then the map $\langle X, K(G,1)\rangle \to H^1(K(G,1))$ sending a map $f : X \to K(G,1)$ to the induced homomorphism $f_* : H_1(X) \to H_1(K(G,1)) \cong G$ is a bijection, where we identify $H^1(X;G)$ with $\mathrm{Hom}(H_1(X), G)$ via the universal coefficient theorem.

Solution. For every map $f : X \to K(G,1)$, it induces a homomorphism $f_* : H_1(X) \to H_1(K(G,1))$, and if g is homotopic to f, they induce the same homomorphism. Thus we get a map

$$\varphi : \langle X, K(G,1)\rangle \to \mathrm{Hom}(H_1(X), H_1(K(G,1))) \cong H^1(K(G,1)).$$

The last identity comes from the fact that $\mathrm{Ext}(H_0(X);\mathbb{Z}) = 0$. Conversely, since G is abelian, a homomorphism $\tilde{h} : H_1(X) \to G$ is induced by a unique $h : \pi_1(X) \to G \cong \pi_1(K(G,1))$ since $H_1(X)$ is the abelianization of $\pi_1(K(G,1))$. By proposition 1.B.9., this homomorphism is induced by a continuous map g. Since all such maps are unique up to homotopy, we have a map

$$\psi : \mathrm{Hom}(H_1(X), H_1(K(G,1))) \to \langle X, K(G,1)\rangle.$$

The definition of ψ tells us that φ and ψ are inverse, hence we are done. ☐

3.2 Cup Product

Exercise 3.2.1. Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori.

Solution. The quotient is taking M_g/A where A is the subspace of 1-cells connecting the base point so that they split the tori.

First we know that

$$H_i(M_g) = \begin{cases} \mathbb{Z} & i = 0, 2, \\ \mathbb{Z}^{2g} & i = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_i(\bigvee_{i=1}^{g} S^1 \times S^1) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}^{2g} & i = 1, \\ \mathbb{Z}^{g} & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the quotient does not change the generators of the 1 cycles, hence q induces an isomorphism between $H_1(M_g)$ and $H_1(\bigvee_{i=1}^{g} S^1 \times S^1)$. Therefore we denote all the generators by a_i, b_i where $i = 1, \cdots, g$. After applying the quotient, the generator of $H_2(M_g)$ becomes the sum of generators of $H_2(\bigvee_{i=1}^{g} S^1 \times S^1)$, hence we know the induced map $q_* : H_i(M_g) \to H_i(\bigvee_{i=1}^{g} S^1 \times S^1)$. Denote the generator of $H_2(M_g)$ by e, and denote the generators of $H_2(\bigvee_{i=1}^{g} S^1 \times S^1)$ by $e_1, \cdots, e_g$.

By the naturality

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(M_g);G) \arrow[r] & H^n(M_g;G) \arrow[r, "h"] & \mathrm{Hom}(H_n(M_g),G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(\bigvee_{i=1}^{g} S^1 \times S^1);G) \arrow[r] \arrow[u, "(q_*)^*"] & H^n(\bigvee_{i=1}^{g} S^1 \times S^1;G) \arrow[r, "h"] \arrow[u, "q^*"] & \mathrm{Hom}(H_n(\bigvee_{i=1}^{g} S^1 \times S^1),G) \arrow[r] \arrow[u, "(q_*)^*"] & 0.
\end{tikzcd}$$

But since that all extension groups vanish, hence q^* maps α_i to α_i, β_i to β_i and ϵ_i to ϵ, where $i = 1, \cdots, g$ and $\alpha_i, \beta_i, \epsilon_i, \epsilon$ are the dual basis of a_i, b_i, e_i, e. Thus the only nontrivial cup product is

$$H_1(M_g) \times H_1(M_g) \xrightarrow{\smile} H_2(M_g).$$

Hence

$$\begin{aligned}\alpha_i \smile \beta_j = q^*(\alpha_i) \smile q^*(\beta_j) &= q^*(\alpha_i \smile \beta_j) \\ &= \begin{cases} q^*(\epsilon_i) & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \\ &= \begin{cases} \epsilon & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}\end{aligned}$$

because of the cup product structure on $S^1 \times S^1$ and a_i and b_j are in different tori when $i \neq j$. Similarly

$$\beta_i \smile \beta_j = q^*(\beta_i) \smile q^*(\beta_j) = q^*(\beta_i \smile \beta_j) = 0$$

and

$$\alpha_i \smile \alpha_j = q^*(\alpha_i) \smile q^*(\alpha_j) = q^*(\alpha_i \smile \alpha_j) = 0$$

for all i and j. □

Exercise 3.2.2 (Exercise 3.2.2)**.** Using the cup product $H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^\bullet(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is the union of n contractible open subsets, to show that all n-fold cup products of positive-dimensional classes are zero.

Solution. We have S.E.S.'s

$$0 \to C^n(X, A; R) \xrightarrow{q_1^n} C^n(X; R) \xrightarrow{i_1^n} C^n(A; R) \to 0$$

and

$$0 \to C^n(X, B; R) \xrightarrow{q_2^n} C^n(X; R) \xrightarrow{i_2^n} C^n(B; R) \to 0,$$

which induce long exact sequences

$$\cdots \to H^n(X, A; R) \to H^n(X; R) \to H^n(A; R) \to H^{n+1}(X, A; R) \to \cdots$$

and

$$\cdots \to H^n(X, B; R) \to H^n(X; R) \to H^n(B; R) \to H^{n+1}(X, B; R) \to \cdots .$$

Since A and B are contractible, we know that $H^n(A; R) = H^n(B; R) = 0$ when $n > 0$. Thus for any two cochains $\varphi \in H^n(X; R)$ and $\psi \in H^n(X; R)$, there are $\alpha \in H^n(X, A; R)$ and $\beta \in H^n(X, B; R)$ s.t. $q_1^*(\alpha) = \varphi$ and $q_2^*(\beta) = \psi$, hence

$$\varphi \smile \psi = q_1^*(\alpha) \smile q_2^*(\beta) = \alpha \smile \beta = 0$$

since $\alpha \smile \beta \in H^{k+l}(X, A \cup B; R) = 0$.

When $X = SY$ is the suspension, then we have a natural structure of $A \simeq \{*\}$ and $B \simeq \{*\}$, so we are done.

Assume now X is covered by n contractible open sets $U_1, \cdots, U_n$, by the same reason the quotient $q_i : C_n(X) \to C_n(X)/C_n(U_i)$ induces an surjections $H^n(X, U_i; R) \twoheadrightarrow H^n(X; R)$ for $n > 0$. Thus for $\varphi_i \in H^n(X; R)$, we have $\alpha_i \in H^n(X, U_i; R)$ s.t. $q_i^*(\alpha_i) = \varphi_i$. Hence

$$\varphi_1 \smile \cdots \smile \varphi_n = q_1^*(\alpha_1) \smile \cdots \smile q_n^*(\alpha_n) = \alpha_1 \smile \cdots \smile \alpha_n = 0.$$

The reasons why $q_1^*(\alpha) \smile q_2^*(\beta) = \alpha \smile \beta$ is as follows: there is a commutative diagram

$$\begin{array}{ccccc} H^k(X, A; R) & \times & H^l(X, B; R) & \xrightarrow{\smile} & H^{k+l}(X, A \cup B; R) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*=0} \\ H^k(X; R) & \times & H^l(X; R) & \xrightarrow{\smile} & H^{k+l}(X; R) \end{array}$$

which says the result, and the commutativity comes from

$$\alpha \smile \beta([\sigma]) :=$$

for any σ □

Exercise 3.2.3 (Exercise 3.2.3)**.** 1. Using the cup product structure, show there is no map $\mathbb{RP}^n \to \mathbb{RP}^m$ inducing a nontrivial map $H^1(\mathbb{RP}^m;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ if $n > m$. What is the corresponding result for maps $\mathbb{CP}^n \to \mathbb{CP}^m$.

2. Prove the Borsuk-Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x. Then define $g : S^n \to S^{n-1}$ by $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{RP}^n \to \mathbb{RP}^{n-1}$. Show that part 1 applies to this map.

Solution. (i) Recall that $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{n+1})$, with $|\alpha| = 1$. If a map $f : \mathbb{RP}^n \to \mathbb{RP}^m$ induced a nontrivial map $f^* : H^1(\mathbb{RP}^m;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$, then $f^*(\alpha) = \alpha$ in the polynomial rings. Therefore,

$$f^*(\alpha^{m+1}) = f^*(\alpha^m)f^*(\alpha) = \alpha^m\alpha = \alpha^{m+1}$$

which is nontrivial in $(\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{m+1})$, a contradiction.

Similarly, since $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, hence the result is that there is no map $\mathbb{CP}^n \to \mathbb{CP}^m$ inducing a nontrivial map $H^1(\mathbb{CP}^m) \to H^1(\mathbb{CP}^n)$ if $n > m$.

(ii) The Borsuk-Ulam theorem states that for any map $f : S^n \to \mathbb{R}^n$, there must be a point $x \in S^n$ s.t. $f(x) = f(-x)$. Otherwise one has an induced map as constructed

$$g : \mathbb{RP}^n \to \mathbb{RP}^{n-1}.$$

Part one says $g^* : H^1(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ is trivial. Thus, to derive the contradiction it suffices to prove that $g^* : H^1(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ is nontrivial.

Take a path in $\mathbb{RP}^n$ representing a generator of $\pi_1(\mathbb{RP}^n)$, then it lifts to a path connecting the base point p and the antipode $-p$. Thus the image of this path under g is the generator of $\pi_1(\mathbb{RP}^{n-1})$, hence g is nontrivial $\pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^{n-1})$. But H_1 is just the abelianization of π_1, where in this case they are isomorphic. Hence g_* is nontrivial $H_1(\mathbb{RP}^n) \to H_1(\mathbb{RP}^{n-1})$ since $g(-x) = -g(x)$. By the naturality and the property of $\mathrm{Ext}^n_{\mathbb{Z}/2\mathbb{Z}}(-,\mathbb{Z}/2\mathbb{Z})$, $H^1 \cong H_1$, hence g induces a nontrivial map $g^* : H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z})$. □

Exercise 3.2.4. Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point if n is even, using the fact that $f^* : H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ is a ring homomorphism. When n is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

Solution. We know that with the cup product structure

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where $\deg\alpha = 2$. If $f^*(\alpha) = k\alpha$ for some integer k (we have this because f^* is a map $H^2(\mathbb{CP}^n;\mathbb{Z}) \to H^2(\mathbb{CP}^n;\mathbb{Z})$). By the cup product structure

$$f^*(\alpha^i) = f^*(\alpha)^i = k^i\alpha^i$$

Since α^i is the generator of $H^i(\mathbb{CP}^n;\mathbb{Z})$, the Lefschetz number is just

$$\tau(f) = \sum_{i=0}^{n}(-1)^{2i}k^i = \sum_{i=0}^{n}k^i.$$

Thus to prove that $\tau(f) \neq 0$ implies that f has fixed point.

If n is even, say $n = 2d$, then notice that $|k^i| \leq |k^{i+1}|$ and if $i+1$ is even, $|k^i| \leq |k^{i+1}| = k^{i+1}$. Thus

$$\tau(f) = \sum_{i=0}^{n}k^i = 1 + \sum_{i=1}^{d}(k^{2i} + k^{2i-1}) \geq 1,$$

and that f has fixed point. If n is odd, we know that f does not admit any fixed point only if

$$\tau(f) = \sum_{i=0}^{n}k^i = 0,$$

and this happens only if $k < 0$. But if $k < -1$, we know that

$$\tau(f) = \sum_{i=0}^{n-1} k^i + k^n \geq 1 + \sum_{i=1}^{\frac{n-1}{2}} k^{2(i-1)}(k^2 + k) + k^n \geq 1,$$

hence there is a fixed point unless $f^*(\alpha) = -\alpha$. □

Exercise 3.2.5 (Exercise 3.2.6)**.** Use cup products to compute the map $H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ induced by the map $\mathbb{CP}^n \to \mathbb{CP}^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to d-th power, $(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d)$, for a fixed integer $d > 0$.

Solution. We know that $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ where $|\alpha| = 2$. By Exercise 2.2.8, if $n = 2$ we know that the map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ s.t. $[z_0; z_1] \mapsto [z_0^d; z_1^d]$ is of degree d, hence by the naturality of Ext,

$$\begin{CD}
0 @>>> \operatorname{Ext}(H_1(\mathbb{CP}^2);G) @>>> H^2(\mathbb{CP}^2) @>h>> \operatorname{Hom}(H_2(\mathbb{CP}^2),\mathbb{Z}) @>>> 0\\
@. @VV(f_*)^*V @VVf^*V @VV(f_*)^*V @.\\
0 @>>> \operatorname{Ext}(H_1(\mathbb{CP}^2);G) @>>> H^2(\mathbb{CP}^2) @>h>> \operatorname{Hom}(H_2(\mathbb{CP}^2),\mathbb{Z}) @>>> 0.
\end{CD}$$

But the extension group are all 0 and $(f_*)^*$ is the multiplication by d since for any $\varphi \in \operatorname{Hom}(H_2(\mathbb{CP}^2),\mathbb{Z})$ that $(f_*)^*(\varphi) = \varphi \circ f_* = d\varphi$. Therefore f^* is the multiplication by d.

Then for any arbitrary n we have commutative diagram

$$\begin{CD}
H^2(\mathbb{CP}^n) @>f^*>> H^2(\mathbb{CP}^n)\\
@VV\mathrm{id}V @VV\mathrm{id}V\\
H^2(\mathbb{CP}^2) @>F^*>> H^2(\mathbb{CP}^2)
\end{CD}$$

where $F : \mathbb{CP}^2 \to \mathbb{CP}^2$ is defined by $[z_0; z_1; z_2] \mapsto [z_0^d; z_1^d; z_n^d]$ and we know that F^* is the multiplication by d. But f restricted on the 0 and 2 cells are exactly F and hence hence the diagram commutes, therefore we have $f^*(\alpha) = d\alpha$. By the cup product, we have

$$f^*(p(\alpha)) = p(d\alpha)$$

for all $p(x) \in H^*(\mathbb{CP}^n;\mathbb{Z})$. □

Exercise 3.2.6 (3.2.7 (Also a proof without using cohomology))**.** Use cup products to show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

Solution. We know that

$$H^*(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^4).$$

And similarly, we also have that

$$H_n(\mathbb{RP}^2 \vee S^3, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{when } n = 0,1,2,3, \\ 0 & \text{otherwise.} \end{cases}.$$

Consider the product of generators of $H^1(\mathbb{RP}^3,\mathbb{Z}/2\mathbb{Z})$ and $H^2(\mathbb{RP}^3,\mathbb{Z}/2\mathbb{Z})$ is the generator of $H^3(\mathbb{RP}^3,\mathbb{Z}/2\mathbb{Z})$,which is just $\alpha\alpha^2 = \alpha^3$. However, the product of generators of $H^1(\mathbb{RP}^2 \vee S^3,\mathbb{Z}/2\mathbb{Z})$ and $H^2(\mathbb{RP}^2 \vee S^3,\mathbb{Z}/2\mathbb{Z})$ is the generator of $H^3(\mathbb{RP}^2,\mathbb{Z}/2\mathbb{Z}) = 0$, because the 3-cell of $\mathbb{RP}^2 \vee S^3$ is only connected to the basepoint, hence they have different cup product. Therefore, two spaces are not homotopy equivalent.

To prove that they are not homotopy equivalent without cohomology, we consider the covering spaces of the two spaces, where the universal covering of $\mathbb{RP}^3$ is S^3, and the universal covering of $\mathbb{RP}^2 \vee S^3$ is $S^3 \vee S^2 \vee S^3$. If they are homotopy equivalent, then they must have the same universal covering space. But $H_3(S^3 \vee S^2 \vee S^3) = \mathbb{Z}^2$, and $H_3(S^3) = \mathbb{Z}$, which is a contradiction. □

Exercise 3.2.7 (3.2.11)**.** Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.

Solution. It is known that

$$H^i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

For any map $f : S^{k+l} \to S^k \times S^l$, it induces the following diagram

$$\begin{array}{ccccc} H^i(S^k \times S^l) & \times & H^j(S^k \times S^l) & \xrightarrow{\smile} & H^{i+j}(S^k \times S^l) \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f^{i+j}} \\ H^i(S^{k+l}) & \times & H^j(S^{k+l}) & \xrightarrow{\smile} & H^{i+j}(S^{k+l}). \end{array}$$

When $i = k$ and $j = l$, $H^i(S^{k+l}) = H^j(S^{k+l}) = 0$, hence by the commutativity, f^{i+j} is trivial. Then by the naturality of the universal coefficient theorem

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}(H^{k+l+1}(S^k \times S^l); \mathbb{Z}) & \longrightarrow & H^{k+l}(S^k \times S^l) & \xrightarrow{h} & \operatorname{Hom}(H_{k+l}(S^k \times S^l), \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (f_*)^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f_*)^*} & & \\ 0 & \longrightarrow & \operatorname{Ext}(H_{k+l-1}(S^{k+l}); \mathbb{Z}) & \longrightarrow & H^{k+l}(S^{k+l}) & \xrightarrow{h} & \operatorname{Hom}(H_{k+l}(S^{k+l}), \mathbb{Z}) & \longrightarrow & 0, \end{array}$$

where all Ext's can be computed 0, so all h are isomorphisms, since f^* is trivial, $(f_*)^*$ is trivial by commutativity. Therefore f_* is trivial. □

3.3 Poincare Duality

Exercise 3.3.1.

Solution. □

Exercise 3.3.2.

Solution. □

Exercise 3.3.3. Show that every covering space of an orientable manifold is an orientable manifold.

Solution. By dealing with each component, one can assume that the manifold M of dimension n is connected. Therefore M is path-connected. Suppose $p : \tilde{M} \to M$ be the covering map. Since M is orientable, there is an orientation

$$\mu : M \to \mathbb{Z} \cong H_n(M, M - \{x\})$$

s.t. it satisfies the local consistency. Thus one can find an open neighbourhood U_x of $x \in M$ s.t. $p^{-1}(U_x)$ consists of disjoint union of open sets homeomorphic to U_x. For any $y \in \tilde{M}$, consider the local homeomorphism

$$p|_{U_y} : U_y \xrightarrow{\sim} U_{p(y)}$$

where U_y is the local neighbourhood homeomorphic to $U_{p(y)}$. Denote $x = p(y)$. It induces a natural isomorphism

$$H_n(\tilde{M}, \tilde{M} - y) \cong H_n(U_{\{y\}}, U_y - \{y\}) \xrightarrow{(p|_{U_y})_*} H_n(U_x, U_x - \{x\}) \cong H_n(M, M - \{x\}).$$

One can construct an assignment

$$\begin{aligned} \tilde{\mu} : \tilde{M} &\to \mathbb{Z} \cong H_n(\tilde{M}, \tilde{M} - \{y\}) \\ y &\mapsto (p|_{U_y})_*^{-1}(\mu_{p(y)}). \end{aligned}$$

By the assumption, for all point $z \in U_x$, the local consistency says μ_z is the same as the one given by μ_x. By the construction, for all point $\tilde{z} \in U_y$, the local consistency says $\mu_{\tilde{z}}$ is the same as the one given by μ_y. Thus one has the local consistency for $\tilde{M}$, so $\tilde{M}$ is also orientable. □

Exercise 3.3.4. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

Solution. Similar to the previous problem, one can assume that the manifold is connected. Let M be the orientable, path connected n-dimensional manifold, and

$$p : M \to M/G$$

be the covering map. Since M is orientable, there is an orientation

$$\mu : M \to \mathbb{Z} \cong H_n(M, M - \{x\})$$

s.t. it satisfies the local consistency. Thus one can find an open neighbourbood U_x of $x \in M$ s.t. $p(U_x)$ is homeomorphic to U_x. The covering map induces a natural isomorphism

$$H_n(M, M - x) \cong H_n(U_x, U_x - \{x\}) \xrightarrow{(p|_{U_x})_*} H_n(p(U_x), p(U_x) - \{p(x)\}) \cong H_n(M/G, M/G - \{p(x)\}).$$

One can construct an assignment

$$\begin{aligned} \bar{\mu} : M/G \to \mathbb{Z} &\cong H_n(M/G, M/G - \{p(x)\}) \\ x &\mapsto (p|_{U_x})_*(\mu_x). \end{aligned}$$

Since each element in G preserves the orientation, the assignment is well-defined, and it automatically satisfies the local consistency. $\square$

Exercise 3.3.5 (3.3.10)**.** Show that for a degree 1 map $f : M \to N$ of connected closed orientable manifolds, the induced map $f_* : \pi_1(M) \to \pi_1(N)$ is surjective, hence also $f_* : H_1(M) \to H_1(N)$.

Solution. We know that there is a covering space $\tilde{N}$ of N with $\pi_1(\tilde{N}) = \mathrm{Im}(f_*)$, and by the lifting property we have the following factorization

$$\begin{tikzcd} M \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \tilde{N} \arrow[d, "p"] \\ & N. \end{tikzcd}$$

Problem 3.3.8. tells us that the degree is multiplicative, hence $1 = \deg\ \tilde{f} \deg\ p$. But we have to take degrees as integers, hence $\deg\ p = \pm 1$. This implies that the covering is 1-sheeted, hence the induced homomorphism is surjective.

Furthermore, we have a map $\pi_1(M) \to \pi_1(N) \xrightarrow{\text{ab}} H_1(N)$, where $H_1(N)$ is the abelianization of $\pi_1(N)$. Thus the map factors through $H_1(M)$ because $H_1(M)$ is the abelianization of $\pi_1(M)$. But the whole map is surjective, therefore the induced map $H_1(M) \to H_1(N)$ is surjective. $\square$

Exercise 3.3.6 (3.3.11)**.** If M_g denotes the closed orientable surface of genus g, show that degree 1 maps $M_g \to M_h$ exist iff $g \geq h$.

Solution. We know that

$$H_n(M_g) = \begin{cases} \mathbb{Z} & \text{when } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases},$$

thus by universal coefficient theorem (or duality)

$$H^n(M_g) = \begin{cases} \mathbb{Z} & \text{when } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{when } n = 1 \\ 0 & \text{otherwise.} \end{cases}.$$

It suffices to prove that a degree 1 map induces a surjective homomorphism $H_1(M_g) \to H_1(M_h)$, hence by the fundamental theorem of modules over P.I.D., $g \geq h$. By previous problem, we already have a surjective homomorphism $H_1(M_g) \to H_1(M_h)$.

Conversely, if $g \geq h$, consider the cell structures of both manifolds, with the orientations s.t. both 2-cells are the positive generators. Hence the quotient map $M_g \to M_g/X \cong M_h$ where $X = \bigcup_{i=h+1}^{g}(a_i \cup b_i)$ satisfies that mapping the 2-cell of M_g to the 2-cell of M_h, and mapping the 1-cells $a_1, b_1, \cdots, a_h, b_h$ to $c_1, d_1, \cdots, c_h, d_h$. Hence it suffices to prove that this map is of degree 1. But the 2-cells are exactly the fundamental class for M_g and M_h, hence this map is of degree 1. □

Exercise 3.3.7 (3.3.17)**.** Show that a direct limit of exact sequences is exact. More generally, show that homology commutes with direct limits: If $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes, with the maps $f_{\alpha\beta} : C_\alpha \to C_\beta$ chain maps, then $H_n(\lim_\to C_\alpha) = \lim_\to H_n(C_\alpha)$.

Solution. By the definition of direct limit, there exist maps $f_\alpha : G_\alpha \to G = \lim_\to G_\alpha$ which are compatible with the direct system, where $f_\alpha(x_\alpha)$ is the image of the quotient. We first prove the following

Lemma 1. *For any element $x \in G$, there is some $\alpha \in I$ s.t. $x = f_\alpha(x_\alpha)$ where $x_\alpha \in G_\alpha$. And $f_\alpha(x_\alpha) = 0$ if and only if there is some $\beta \geq \alpha$ s.t. $f_{\alpha\beta}(x_\alpha) = 0$.*

Proof. For any $x \in G = \bigoplus_{\alpha\in I} G_\alpha/N$ where N is generated by all the elements $a - f_{\alpha\beta}(a)$, we know

$$x = \sum_{\alpha\in I} x_\alpha + N$$

where the sum is finite. Suppose the sum is taken as $x = \sum_{i=1}^{n} x_{\alpha_i} + N$ and we have a β s.t. $\alpha_i \leq \beta$, thus

$$\begin{aligned} x &= \sum_{i=1}^{n} x_{\alpha_i} + N \\ &= \sum_{i=1}^{n} x_{\alpha_i} + \sum_{i=1}^{n}(f_{\alpha_i\beta}(x_{\alpha_i}) - x_{\alpha_i}) + N \\ &= \sum_{i=1}^{n} f_{\alpha_i\beta}(x_{\alpha_i}) + N, \end{aligned}$$

where this is saying x is $f_\beta(\sum_{i=1}^{n} f_{\alpha_i\beta}(x_{\alpha_i}))$.

For the other part, $f_{\alpha\beta}(x_\alpha) = 0$ means the image of x_α in the direct sum is in N. Thus there is some $\beta \geq \alpha$ s.t. $x_\alpha = x_\alpha - f_{\alpha\beta}(x_\alpha)$. But this really means that $f_{\alpha\beta}(x_\alpha) = 0$. Conversely, if $f_{\alpha\beta}(x_\alpha) = 0$ for some $\beta \geq \alpha$, then $x_\alpha = x_\alpha - f_{\alpha\beta}(x_\alpha) \in N$, hence $f_\alpha(x_\alpha) = 0$. □

Back to the problem, since $H_n(\lim_\to C_\alpha) = \mathrm{Ker}\ d_n/\mathrm{Im}\ d_{n+1}$, it suffices to prove that if for each $\alpha \in I$, the map $d_\alpha : C_\alpha \to C'_\alpha$ is compatible with the direct system (the compatibility comes from $f_{\alpha\beta}$ being chain maps), then $\mathrm{Ker}(\lim_\to d_\alpha) = \lim_\to \mathrm{Ker}(d_\alpha)$ and $\mathrm{Im}(\lim_\to d_\alpha) = \lim_\to \mathrm{Im}(d_\alpha)$. Just by the functoriality of direct limit, we have a canonical map $d = \lim_\to d_\alpha : \lim_\to C_\alpha \to \lim_\to D_\alpha$.

If $x = x_\alpha + N \in \mathrm{Ker}(d)$ by the lemma, then $d_\alpha(x_\alpha) \in N'$. This means that x_α lives in $\mathrm{Ker}(d_\alpha)$, therefore $x \in \lim_\to \mathrm{Ker}(d_\alpha)$. Conversely, if $x \in \lim_\to \mathrm{Ker}(d_\alpha)$, then there is some α s.t. $x = x_\alpha$ with $d_\alpha(x_\alpha) + N' = 0$. Hence $x = x_\alpha + N \in \mathrm{Ker}(d)$.

On the other hand, if $y = y_\alpha + N' \in \mathrm{Im}(\lim_\to d_\alpha)$, then by the lemma $y_\alpha \in \mathrm{Im}(d_\alpha)$, hence $x \in \lim_\to \mathrm{Im}(d_\alpha)$. Conversely, if $x = y_\alpha + N' \in \lim_\to \mathrm{Im}(d_\alpha)$, we find the α s.t. $y_\alpha \in \mathrm{Im}(d_\alpha)$ by the lemma. Hence we are done. □

Exercise 3.3.8 (3.3.18)**.** Show that a direct limit $\lim_\to G_\alpha$ of torsion-free abelian groups is torsion-free. More generally, show that any finitely generated subgroup of $\lim_\to G_\alpha$ is realized as a subgroup of some G_α.

Solution. It suffices to prove the second part of the problem. Suppose a subgroup of $G := \lim_\to G_\alpha$ is generated by $x_1, \cdots, x_n$. By the lemma proved before, there are some $\alpha_1, \cdots, \alpha_n$ s.t. $x_i = f_{\alpha_i}(x_{\alpha_i})$ for some $x_{\alpha_i} \in G_{\alpha_i}$, then for any $\beta \geq \alpha_i$ we have the images of $f_{\alpha_i\beta}(x_{\alpha_i})$ in G generates the subgroup.

Suppose that for some generator x of the subgroup, there is some integer $n > 0$ s.t. $nx = 0$ but $kx \neq 0$ for all $0 < k < n$. By previous lemma, we have some $x_\alpha \in G_\alpha$ with $f_\alpha(x_\alpha) = x$. Thus

$$0 = nx = nf_\alpha(x_\alpha) = f_\alpha(nx_\alpha),$$

which means there is a $\beta \geq \alpha$ s.t. $nf_{\alpha\beta}(x_\alpha) = 0$. But $kx \neq 0$ means $kf_{\alpha\beta}(x_\alpha) = f_{\alpha\beta}(kx_\alpha) \neq 0$, hence kx_α for any $\beta \geq \alpha$. Therefore there is a sufficiently large γ s.t. the subgroup is a subgroup of G_γ. □

Exercise 3.3.9 (3.3.25)**.** Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsion-free, then $H_k(M;\mathbb{Z})$ is also torsion-free.

Solution. By the universal coefficient theorem we have

$$0 \to \mathrm{Ext}(H_{k-1}(M,\mathbb{Z}), Z) \to H^k(M,\mathbb{Z}) \to \mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{Z}) \to 0$$

and $H_{k-1}(M,\mathbb{Z})$ being tells that the Ext term is zero so we know that $H^k(M,\mathbb{Z}) \cong \mathrm{Ext}(H_k(M,\mathbb{Z}),\mathbb{Z})$. Now since M is a closed orientable manifold we can apply Poincare duality which shows that

$$H_k(M,\mathbb{Z}) \cong H^{2k-k}(M,\mathbb{Z}) = H^k(M,\mathbb{Z}).$$

This together with our previous statement show that $H_k(M,\mathbb{Z}) \cong \mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{Z})$. Now assume that $H_K(M,\mathbb{Z})$ is has torsion then since there are no homomorphisms (except for the trivial homomorphism) from a group of finite order into $\mathbb{Z}$ we know that $\mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{Z})$ is torsion free which is a contradiction. □

Exercise 3.3.10 (3.3.8)**.**

Solution. We shall use the commutative diagram in Page 136, of proposition 2.30, but replacing S^n by M and N. Pick one $y \in B$. By excision, the central term $H_n(M, M - f^{-1}(y))$ is the direct sum of groups $H_n(B_i, B_i - x_i) \cong \mathbb{Z}$, with k_i the inclusion of i-th summand and p_i the projection of i-th summand. The commutativity of the lower triangle says that $p_i j(1) = 1$, hence $j(1) = \sum_i k_i(1)$. But commutativity of upper square says that the middle f_* takes $k_i(1)$ to the local degree, hence take the sum $\sum_i k_i(1)$ to $\sum_i \deg\ f|_{x_i} = \sum_i \epsilon_i$. The commutativity of lower square then gives what we want.

An immediate corollary is 3.3.9., where if π is a p-sheeted covering space, then $\epsilon_1 = \cdots = \epsilon_p = \pm 1$, hence the degree of π is $\pm p$. Also, if $p = \infty$, the argument also works. □

Exercise 3.3.11 (3.3.8)**.** Let l_1, l_2, l_3 be three projective lines in $\mathbb{CP}^2$ such that $l_1 \cap l_2 \cap l_3 = \emptyset$. Let L_i be the compact tube neighborhood of l_i in $\mathbb{CP}^2$, with that $W = L_1 \cup L_2 \cup L_3$ is a compact (real) 4-dimensional manifold with boundaries. If $W - l_1 \cup l_2 \cup l_3 \cong \partial W \times [0,1)$, compute $H_i(\partial W;\mathbb{Z})$.

Solution. □

Chapter 4

Homotopy Theory

4.1 Homotopy Groups

Exercise 4.1.1. Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]

Solution. □