Math 4500 HW #01 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 1.2.4 (5 pts))**.** Deduce, from the distributive law and multiplicative absolute value, that

$$|uv - uw| = |u||v - w|.$$

Explain why this says that multiplication of the whole plane of complex numbers by u multiplies all distances by $|u|$.

Solution.

$$|uv - uw| = |u(v - w)| = |u||v - w|.^{[2]}$$

Suppose v, w are two points in the complex plane, then $|v - w|$ represents the distance between two points and $|uv - uw|$ represents the distance between points after the transformation that multiplying the whole plane by u.[2] The equation means that the distance changes with a parameter $|u|$.[1] □

Problem 2 (Exercise 1.2.5 (5 pts))**.** Deduce from Exercise 1.2.4 that multiplication of the whole plane of complex numbers by $\cos\theta + i\sin\theta$ leaves all distances unchanged.

Solution. Denote $u = \cos\theta + i\sin\theta$ then $|u| = \cos^2\theta + \sin^2\theta = 1$.[2] Suppose v, w are two points in the complex plane, then by previous exercise $|uv - uw| = |u||v - w| = |v - w|$,[1] where $|v - w|$ represents the distance between previous points and $|uv - uw|$ means the distance between the new points. Hence the transformation leaves all distances unchanged.[2] □

Problem 3 (Exercise 1.3.6 (10 pts))**.** Show that the multiplicative property of determinants gives the *real four-square identity*

$$(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)^2$$
$$+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2 + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)^2$$

Solution. Let $\alpha := \begin{pmatrix} a_1 + id_1 & -b_1 - ic_1 \\ b_1 - ic_1 & a_1 - id_1 \end{pmatrix}$ and $\beta := \begin{pmatrix} a_2 + id_2 & -b_2 - ic_2 \\ b_2 - ic_2 & a_2 - id_2 \end{pmatrix}$. Then the determinants gives

$$|\alpha||\beta| = |\alpha\beta|,^{[3]}$$

where the left hand side is just

$$(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2).^{[2]}$$

On the other hand, the right hand side is

$$|\alpha\beta| = \det\left(\begin{pmatrix} a_1 + id_1 & -b_1 - ic_1 \\ b_1 - ic_1 & a_1 - id_1 \end{pmatrix}\begin{pmatrix} a_2 + id_2 & -b_2 - ic_2 \\ b_2 - ic_2 & a_2 - id_2 \end{pmatrix}\right)$$
$$= \det\begin{pmatrix} (a_1 + id_1)(a_2 + id_2) + (-b_1 - ic_1)(b_2 - ic_2) & (a_1 + id_1)(-b_2 - ic_2) + (-b_1 - ic_1)(a_2 - id_2) \\ (a_1 + id_1)(b_1 - ic_1) + (a_1 - id_1)(b_2 - ic_2) & (-b_2 - ic_2)(b_1 - ic_1) + (a_1 - id_1)(a_2 - id_2) \end{pmatrix}$$
$$= \det\begin{pmatrix} (a_1a_2 - a_1d_2 - b_1b_2 - c_1c_2) + i(a_2d_1 + a_1d_2 + c_2b_1 - c_1b_2) & -(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) - i(a_1c_2 - b_1 \\ -(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) + i(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) & (a_1a_2 - a_1d_2 - b_1b_2 - c_1c_2) + i(a_2d_1 + a_1 \end{pmatrix}$$

$= RHS$.[5]

□

Problem 4 (Exercise 1.4.4 (10 pts))**.** Also deduce the *Jacobi identity* for the cross product:

$$u \times (v \times w) + w \times (u \times v) + v \times (w \times u) = 0.$$

The antisymmetric and Jacobi properties show that the cross product is not completely lawless. These properties define what we later call a *Lie algebra*.

Solution. We first verify the identity

$$u \times (v \times w) = (w \cdot u)v - (u \cdot v)w.$$

Suppose that $u = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $w = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. By definition, we have

$$\begin{aligned} u \times (v \times w) &= u \times ((v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\ &= (u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3))\mathbf{i} \\ &\quad + (u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1))\mathbf{j} \\ &\quad + (u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2))\mathbf{k} \\ &= ((u_2w_2 + u_3w_3)v_1 - (u_2v_2 + u_3v_3)w_1)\mathbf{i} \\ &\quad + ((u_3w_3 + u_1w_1)v_2 - (u_3v_3 + u_1v_1)w_2)\mathbf{j} \\ &\quad + ((u_1w_1 + u_2w_2)v_3 - (u_1v_1 + u_2v_2)w_3)\mathbf{k} \\ &= ((u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1)\mathbf{i} \\ &\quad + ((u_2w_2 + u_3w_3 + u_1w_1)v_2 - (u_2v_2 + u_3v_3 + u_1v_1)w_2)\mathbf{j} \\ &\quad + ((u_3w_3 + u_1w_1 + u_2w_2)v_3 - (u_3v_3 + u_1v_1 + u_2v_2)w_3)\mathbf{k} \\ &= (w \cdot u)v - (u \cdot v)w.^{[7]} \end{aligned}$$

Back to the problem,

$$\begin{aligned} &u \times (v \times w) + w \times (u \times v) + v \times (w \times u) \\ =& ((w \cdot u)v - (u \cdot v)w) + ((v \cdot w)u - (w \cdot u)v) + ((u \cdot v)w - (v \cdot w)u) \\ =& 0.^{[3]} \end{aligned}$$

□

Remark. *If your find Exercise 1.5.5 and 1.5.6 hard to guess/prove the answer, go to do Exercise 1.5.1-1.5.4. The 2-dimensional case would tell you everything.*

Problem 5 (Exercise 1.5.5 (20 pts))**.** Adapt the argument of Exercise 1.5.3 to great circles $\mathscr{L}, \mathscr{M}, \mathscr{N}$ shown in the picture. What is the conclusion?

Solution. Literally there are two conclusions: (i) any rotation can be decomposed as a composition of two reflections, and the composition of two reflections is a rotation; (ii) two rotations make a rotation (or all rotations form a group with the multiplication being the composition of maps).[5]

Suppose A is a rotation about a line l through the origin. Let $l \cap S^2 = \{P, Q\}$, and let $\mathscr{M}$ be an ARBITRARY great circle through P and Q. Also we use $\mathscr{M}$ to denote the plane through the line l. Let $\mathscr{L}$ be another great circle through P and Q s.t. the angle between $\mathscr{M}$ and $\mathscr{N}$ is $\theta/2$ where θ is the rotation angle. Let x be an arbitrary point, and let x' be the point of x reflected by the great circle $\mathscr{L}$, and let x'' be the point of x' reflected by the great circle $\mathscr{M}$. Then $\angle xPx'' = 2\frac{\theta}{2} = \theta$, and $|xP| = |x''P|$. Hence A can be decomposed as the composition of reflections, by the great circle $\mathscr{M}$ and by the great circle $\mathscr{M}$ respectively. The same argument proves that the composition of two reflections is a rotation.[10]

Suppose A_1, A_2 are two given rotations, and P, Q are points fixed by A_1, A_2 respectively. As shown in the picture, denote the great circle through points P, Q as $\mathscr{M}$. Let $\mathscr{L}, \mathscr{N}$ are the great circles s.t. the angle from

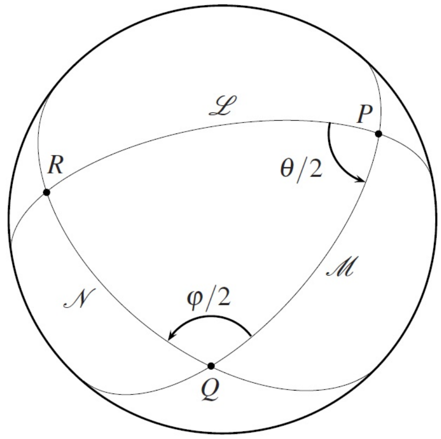

Figure 1: Reflection in great circles on the sphere

$\mathscr{L}$ to $\mathscr{M}$ is $\frac{\theta}{2}$ and the angle from $\mathscr{M}$ to $\mathscr{N}$ is $\frac{\varphi}{2}$, where θ, φ are the angle of the rotations. By abuse of notation, we also denote the reflection by great circles $\mathscr{L}, \mathscr{M}, \mathscr{N}$ as $\mathscr{L}, \mathscr{M}, \mathscr{N}$. Hence by the first conclusion,

$$A_1 = \mathscr{M} \circ \mathscr{L},\ A_2 = \mathscr{N} \circ \mathscr{M}.$$

Therefore

$$A_2 \circ A_1 = \mathscr{N} \circ \mathscr{M} \circ \mathscr{M} \circ \mathscr{L} = \mathscr{N} \circ \mathscr{L},$$

which implies $A_2 \circ A_1$ is again a rotation.[5] □

Problem 6 (Exercise 1.5.6 (5 pts))**.** Explain why there is no exceptional case analogous to Exercise 1.5.4. Deduce that the product of any two rotations of $\mathbb{R}^3$ about O is another rotation, and explain how to find the axis of the product rotation.

Solution. Suppose we have three planes $\mathscr{L}, \mathscr{M}, \mathscr{N}$ in the space going through the origin, then any two of them (w.l.o.g. assume they are $\mathscr{L}$ and $\mathscr{M}$) intersect, so their intersection is a line denoted by l. Suppose l intersects the unit sphere at a point R, then R is an intersection point of the great circles which are the intersections $\mathscr{L} \cap S^2$ and $\mathscr{M} \cap S^2$, where S^2 denotes the sphere.[5] □

Remark. *Generally it is not OK to use some statement that you cannot prove. It is a truth that on a sphere there cannot be parallel geodesics. But it requires a lot to define what is a geodesic and to actually prove the proposition. So if you want to use some fancy things in the homework, explicitly write down everything, every definition and every proof.*

Problem 7 (Exercise 2.1.3 (0 pts))**.** Other than the trivial group $\{1\}$, what is the smalest subgroup of $SO(2)$?

Solution. There are two kinds of elements in $SO(2)$. Some $z \in SO(2)$ satisfy the property that there exists some $m \in \mathbb{Z}$ s.t. $z^m = 1$ while some are not. Those elements with this property are called **of finite order**. To form a subgroup, if $z \in H$, we must have $z^n \in H$ for any $n \in \mathbb{Z}$. Thus, if z is of infinite order, H must be a infinite subgroup. (Actually in this case $\mathbb{Z}$ is a subgroup of H.) There are finite subgroups of $SO(2)$, hence H cannot contain element of infinite order. To make the subgroup smallest, we find that $\{-1, 1\}$ is a nontrivial subgroup and there does not exist nontrivial subgroup whose cardinality is strictly smaller than 2. □

Problem 8 (Exercise 2.1.5 (0 pts))**.** Show that the union R of all the finite subgroups of $SO(2)$ is also a subgroup (the group of "rational rotations").

Solution. Suppose

$$R = \bigcup_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$$

is the group of rational rotations. It suffices to prove that the multiplication is closed. For any $z, w \in R$, we have $z^n = 1$ and $w^m = 1$ for some $n, m \in \mathbb{Z}$. Notice $SO(2)$ is an abelian group so that $(zw)^{lcm(n,m)} = z^{lcm(n,m)} w^{lcm(n,m)} = 1$, which means zw is also an element in R. ☐

Problem 9 (Exercise 2.1.6 (0 pts))**.** If z is a complex number not in the rational rotation group R described in Exercise 2.1.5, show that all the numbers $\cdots, z^{-2}, z^{-1}, 1, z, z^2, \cdots$ are distinct, and that they form a subgroup of $SO(2)$.

Solution. Suppose $z^m = z^n$ for some $n, m \in \mathbb{Z}$, then $z^{m-n} = 1$. But z is not in the rational rotation group, $m - n$ must be 0. It suffices to verify the associativity, but it comes from the multiplicative associativity of complex numbers. ☐