Math 4500 HW #03 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 2.5.4 (5 pts))**.** By the reflection formula, the product

$$q \mapsto u_4\bar{u}_3u_2\bar{u}_1 q \bar{u}_1 u_2 \bar{u}_3 u_4$$

is a reflection in the hyperplanes orthogonal to u_1, u_2, u_3, u_4 respectively. Check that $u_4\bar{u}_3u_2\bar{u}_1 = i$ and $\bar{u}_1u_2\bar{u}_3u_4 = 1$, so the product of the four reflections is indeed $q \mapsto iq$.

Solution. Here we know that $u_1 = i, u_2 = \frac{-1+i}{\sqrt{2}}, u_3 = k, u_4 = \frac{-j+k}{\sqrt{2}}$, hence $\bar{u}_1 = -i, \bar{u}_2 = \frac{-1-i}{\sqrt{2}}, \bar{u}_3 = -k, \bar{u}_4 = \frac{j-k}{\sqrt{2}}$. Thus

$$u_4\bar{u}_3u_2\bar{u}_1 = \frac{-j+k}{\sqrt{2}}(-k)\frac{-1+i}{\sqrt{2}}i = \frac{1+i}{\sqrt{2}}\frac{-1+i}{\sqrt{2}}i = i$$

and

$$\bar{u}_1u_2\bar{u}_3u_4 = -i\frac{-1+i}{\sqrt{2}}(-k)\frac{-j+k}{\sqrt{2}} = \frac{j+k}{\sqrt{2}}\frac{-j+k}{\sqrt{2}} = 1.$$

□

Problem 2 (Exercise 2.7.1 (5 pts))**.** Check that $q \mapsto u^{-1}qu$ is an automorphism of $\mathbb{H}$ for any unit quaternion u.

Solution. It suffices to check that $q \mapsto u^{-1}qu$ is an isomorphism from $\mathbb{H}$ to $\mathbb{H}$.

For any two quaternion q_1, q_2, we have

$$u^{-1}(q_1q_2)u = u^{-1}(q_1(uu^{-1})q_2)u = (u^{-1}q_1u)(u^{-1}q_2u)$$

and

$$u^{-1}(q_1+q_2)u = (u^{-1}q_1 + u^{-1}q_2)u = u^{-1}q_1u + u^{-1}q_2u$$

since the multiplication is a homomorphism. These imply that $q \mapsto u^{-1}qu$ is a homomorphism $\mathbb{H} \to \mathbb{H}$.

If

$$u^{-1}q_1u = u^{-1}q_2u$$

for two quaternion q_1, q_2, then by multiplying u and u^{-1}, we know $q_1 = q_2$ and the map is injective. For any $q \in \mathbb{H}$, the map sends uqu^{-1} to q. Hence it is surjective.[5] The continuity is derived from the fact that matrix multiplication is continuous. □

Problem 3 (Exercise 3.1.2 (5 pts))**.** Give an example of a matrix in $O(3)$ that is not in $SO(3)$, and interpret it geometrically.

Solution. Consider

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

is an element in $O(n)$. But $\det(A) = -1$, which means $A \notin SO(3)$. Geometrically it is the reflection about the xOy plane. □

Problem 4 (Exercise 3.2.1 (10 pts))**.** Bearing in mind that matrix multiplication is a continuous operation, show that if there are continuous paths in G from I to $A \in G$ and to $B \in G$ then there is a continuous path in G from A to AB.

Solution. Since I and B are path-connected, we have a path

$$\alpha : [0,1] \to G$$

s.t. α is continuous and $\alpha(0) = I$, $\alpha(1) = B$. Since the multiplication of matrix is continuous, we have another path

$$\begin{aligned} \bar{\alpha} : [0,1] &\to G \\ t &\mapsto A \cdot \alpha(t). \end{aligned}$$

Since $\bar{\alpha}(0) = A$ and $\bar{\alpha}(1) = AB$, we know $\bar{\alpha}$ is a path connecting A and AB. Hence A and AB are path-connected. □

Remark. *Here it is not necessary to know the path-connectedness of I and B. However, the textbook did not give us the definition of path-connectedness, which made a lot of people confused. The formal definition is in a space X, two points x, y are said to be path-connected if there is a continuous map*

$$\alpha : [0,1] \to X$$

s.t. $\alpha(0) = x$ and $\alpha(1) = y$.

Problem 5 (Exercise 3.2.2 (5 pts))**.** Similarly show that if there is a continuous path in G from I to C, then there is also a continuous path from C^{-1} to I.

Solution. By previous exercise, put $B = C$ and $A = C^{-1}$. □

Problem 6 (Exercise 2.7.2 (0 pts))**.** Prove that an automorphism ρ of $\mathbb{H}$ preserves 0 and differences.

Solution. Consider

$$\rho(0) = \rho(0+0) = \rho(0) + \rho(0),$$

thus $\rho(0) = 0$. For any two $p, q \in \mathbb{H}$, since ρ is an automorphism, $\rho(p) = \rho(p - q + q) = \rho(p - q) + \rho(q)$, thus it preserves the differences. □

Problem 7 (Exercise 2.7.3 (0 pts))**.** Prove that an automorphism ρ of $\mathbb{H}$ preserves 1 and quotients.

Solution. Similar to 2.7.2. □

Problem 8 (Exercise 2.7.4 (0 pts))**.** Prove that an automorphism ρ of $\mathbb{H}$ is a $\mathbb{R}$ linear map.

Solution. First for any $m, n \in \mathbb{Z}$ s.t. $n \neq 0$, we have

$$\rho(mq) = \rho(q + \cdots + q) = \rho(q) + \cdots + \rho(q) = m\rho(q)$$

and

$$n\rho\left(\frac{m}{n}q\right) = \rho\left(n\frac{m}{n}q\right) = \rho\left(mq\right) = m\rho(q),$$

therefore $\rho\left(\frac{m}{n}q\right) = \frac{m}{n}\rho(q)$.

For any real number r, there exists a sequence of rational numbers $\{r_k\}_{k\in\mathbb{N}}$ s.t. $r_k \to r$ as $k \to \infty$. Hence by the continuity of ρ,

$$\rho(rq) = \rho\left(\left(\lim_{k\to+\infty} r_k\right) q\right) = \lim_{k\to+\infty} \rho(r_k q) = \lim_{k\to+\infty} r_k \rho(q) = r\rho(q).$$

□