Math 4500 HW #04 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 3.4.1 (5 pts))**.** It is easy to test whether a matrix consists of blocks of the form

$$\begin{matrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{matrix}.$$

Nevertheless, it is sometimes convenient to describe the property of "being of the form $C(A)$" more algebraically. One way to do this is with the help of the special matrix

$$J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

If $B := \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, show that $JBJ^{-1} = \bar{B}$.

Solution. It is easy to see that $-JJ = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = I$ hence $J^{-1} = -J$. Thus

$$JBJ^{-1} = -\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = -\begin{pmatrix} \bar{\beta} & \bar{\alpha} \\ -\alpha & \beta \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ \beta & \alpha \end{pmatrix}.$$

□

Problem 2 (Exercise 3.4.2 (5 pts))**.** Conversely, show that if $JBJ^{-1} = \bar{B}$ and $B := \begin{pmatrix} c & d \\ e & f \end{pmatrix}$ then we have $\bar{c} = f$ and $\bar{d} = -e$, so B has the form $\begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$.

Solution. Same as the calculation above,

$$JBJ^{-1} = -\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} -e & -f \\ c & d \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} f & -e \\ -d & c \end{pmatrix}.$$

Since $JBJ^{-1} = \bar{B}$, we know

$$\begin{pmatrix} f & -e \\ -d & c \end{pmatrix} = \begin{pmatrix} \bar{c} & \bar{d} \\ \bar{e} & \bar{f} \end{pmatrix}.$$

This is what we want. □

Problem 3 (Exercise 3.4.3 (13 pts))**.** Now suppose that B_{2n} is any $2n \times 2n$ complex matrix, and let

$$J_{2n} := \begin{pmatrix} J & & \cdots & \\ & J & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & J \end{pmatrix}.$$

Use block multiplication, and the results of Exercises 3.4.1 and 3.4.2, to show that B_{2n} has the form $C(A)$ if and only if $J_{2n}B_{2n}J_{2n}^{-1} = \bar{B}_{2n}$.

Solution. Notice that

$$J_{2n}^{-1} := \begin{pmatrix} J^{-1} & & \cdots & \\ & J^{-1} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & J^{-1} \end{pmatrix} = -J_{2n}.$$

because the multiplication and inverse can be calculated by blocks. And also we have

$$\begin{aligned} J_{2n}B_{2n}J_{2n}^{-1} &= \begin{pmatrix} J & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & J \end{pmatrix} \begin{pmatrix} Q_{1,1} & \cdots & Q_{1,n} \\ \vdots & \ddots & \vdots \\ Q_{n,1} & \cdots & Q_{n,n} \end{pmatrix} \begin{pmatrix} -J & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & -J \end{pmatrix} \\ &= \begin{pmatrix} JQ_{1,1} & \cdots & JQ_{1,n} \\ \vdots & \ddots & \vdots \\ JQ_{n,1} & \cdots & JQ_{n,n} \end{pmatrix} \begin{pmatrix} -J & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & -J \end{pmatrix} \\ &= \begin{pmatrix} JQ_{1,1}(-J) & \cdots & JQ_{1,n}(-J) \\ \vdots & \ddots & \vdots \\ JQ_{n,1}(-J) & \cdots & JQ_{n,n}(-J) \end{pmatrix} \end{aligned}$$

By problem 3.4.2, for each block $JQ_{i,j}J^{-1} = \overline{Q_{i,j}}$ iff so $Q_{i,j}$ has the form $\begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, then so is B_{2n}. □

Problem 4 (Exercise 3.4.4 (7 pts))**.** By taking det of both sides of the equation in Exercise 3.4.3, show that $\det(B_{2n})$ is real.

Solution. Since we already know that $J_{2n}B_{2n}J_{2n}^{-1} = \bar{B}_{2n}$, by taking determinant we have $\det(J_{2n})\det(B_{2n})\det(J_{2n}^{-1}) = \det(J_{2n}B_{2n}J_{2n}^{-1}) = \det(\bar{B}_{2n}) = \overline{\det(B_{2n})}$. Notice that $\det(J) = 1$ and the determinant can be calculated by blocks,

$$\det(J_{2n}) = \det(J)^n = 1.$$

Hence we have $\det(B_{2n}) = \overline{\det(B_{2n})}$, which means $\det(B_{2n})$ is real. □

Problem 5 (Exercise 3.4.5 (5 pts))**.** Assuming now that B_{2n} is in the complex form of $Sp(n)$, and hence is unitary, show that $\det(B_{2n}) = \pm 1$.

Solution. Since B_{2n} is in the complex form of $Sp(n)$, hence it is unitary and

$$B_{2n}\overline{B_{2n}^T} = I.$$

Take determinant we have $\det(B_{2n})\overline{\det(B_{2n})} = 1$, but since $\det(B_{2n})$ is real we know $\det(B_{2n}) = \pm 1$. □

Problem 6 (Exercise 3.3.2 (0 pts))**.** Show that vectors from an orthogonal basis of $\mathbb{C}^n$ if and only if their conjugates form an orthogonal basis, where the conjugate of a vector $(u_1, \cdots, u_n)$ is $(\bar{u_1}, \cdots, \bar{u_n})$.

Solution. By the reflexivity of conjugate, it suffices to prove that vectors from an orthogonal basis of $\mathbb{C}^n$ only if their conjugates form an orthogonal basis. Suppose $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ is an orthogonal basis of $\mathbb{C}^n$, then we know $\boldsymbol{v}_i^T\bar{\boldsymbol{v}}_j = \delta_{ij}$. Because δ_{ij} are always real, hence $\delta_{ij} = \bar{\delta_{ij}} = \overline{\boldsymbol{v}_i^T\bar{\boldsymbol{v}}_j} = \bar{\boldsymbol{v}_i}^T\boldsymbol{v}_j$, which means $\{\bar{\boldsymbol{v}}_1, \cdots, \bar{\boldsymbol{v}}_n\}$ is an orthogonal basis of $\mathbb{C}^n$. □

Problem 7 (Exercise 3.3.4 (0 pts))**.** Show that $A\bar{A}^T = I$ if and only if the column vectors of A form an orthogonal basis.

Solution. It is easy to see that $A\bar{A}^T = I$ iff $\bar{A}^TA = I$. Denote the column vectors of A by $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$, then $\bar{A}^TA = I$ implies $\boldsymbol{a}_i^T\bar{\boldsymbol{a}}_j = \delta_{ij}$, which means $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ forms an orthogonal basis. □

Problem 8 (Exercise 3.3.5 (0 pts))**.** Show that if A preserves the Hermitian inner product, then the column vectors form an orthogonal basis.

Solution. We denote the standard basis by $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$, then since matrix A preserve Hermitian product, $\delta_{ij} = \boldsymbol{e}_i^T\bar{\boldsymbol{e}}_j = (A\boldsymbol{e})_i^T\overline{A\boldsymbol{e}_j} = \boldsymbol{a}_i^T\bar{\boldsymbol{a}}_j$. Thus the column vectors form an orthogonal basis. □