Math 4500 HW #05 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (35 pts)**.** Suppose V is a finite dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$), and suppose B is a function $V \times V \to \mathbb{R}$ (or $V \times V \to \mathbb{C}$). We say B is a bilinear form if for any $u, v, w \in V$ and $a \in \mathbb{R}$ (or $\mathbb{C}$)

$$\begin{aligned} B(u+v, w) &= B(u, w) + B(v, w) \\ B(u, v+w) &= B(u, v) + B(u, w) \\ B(au, v) &= aB(u, v) \\ B(u, av) &= \bar{a}B(u, v) \end{aligned}$$

hold. We say a linear form B is symmetric if $B(u, v) = B(v, u)$ for all $u, v \in V$, anti-symmetric if $B(u, v) = -B(v, u)$ for all $u, v \in V$, alternating if $B(u, u) = 0$ for all $u \in V$.

(i) Suppose $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined as $B(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}\boldsymbol{v}^T$ is a symmetric bilinear form over $\mathbb{R}^n$.

(ii) Suppose $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, defined as $H(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}\bar{\boldsymbol{v}}^T$ is a bilinear form over $\mathbb{C}^n$ but not symmetric.

(iii) Suppose ω is an anti-symmetric form defined on $\mathbb{R}^4$ as

$$\begin{aligned} \omega(e_i, e_j) &= 1 \quad \text{if } i < j \\ \omega(e_i, e_i) &= 0 \end{aligned}$$

for all $i, j = 1, \cdots, 4$. Then define group $G := \{A \in GL_4(\mathbb{R}) \mid \omega(Ax, Ay) = \omega(x, y) \text{ for all } x, y \in \mathbb{R}^2\}$. Prove that G is a group.

(iv) Show that for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^4$,

$$B_{3,1}(\boldsymbol{u}, \boldsymbol{v}) = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4$$

is a symmetric bilinear form. Is $B_{3,1}$ an inner product? Define

$$G := \{A \in GL_4(\mathbb{R}) \mid B_{3,1}(Ax, Ay) = B_{3,1}(x, y) \text{ for all } x, y \in \mathbb{R}^2\}.$$

Prove that G is a group. It is denoted by $SO(3, 1)$ and is called the Lorentz group and it plays an important role in physics.

Solution. I leave all the verifications in (i) to (iv) for you to check. They should be easy.

(i) B is a bilinear form.[4]. It is symmetric.[2]

(ii) B is a bilinear form.[4]. It is not symmetric simply because $H(u, v) = i \neq -i = H(v, u)$ where $u = (1, 0, \cdots, 0)$ and $v = (i, 0, \cdots, 0)$.[3]

(iii) ω is antisymmetric since it is antisymmetric on a basis.[4] The reason why there is a subgroup $G \subseteq GL(V)$ does not depend on the structure of the bilinear form. As long as there is a bilinear form ω (we do not have any extra information on the bilinear form), there is a subgroup G of $GL(V)$ defined by

$$G := \{A \in GL(V) \mid B(Ax, Ay) = B(x, y) \text{ for all } x, y \in V\}.$$

To verify it is a group, it suffices to prove G is a subgroup of $GL(V)$. So we need to check that: (a) the identity matrix I is contained in G, (b) if A, B are elements in G, then so is AB, (c) if $A \in G$, then $A^{-1} \in G$.

(a) is clear. For $A, B \in G$, we know that

$$\begin{aligned} \omega(ABx, ABy) &= \omega(A(Bx), A(By)) \\ &= \omega(Bx, By) \\ &= \omega(x, y). \end{aligned}$$

For $A \in G$, then

$$\begin{aligned} \omega(x, y) &= \omega((AA^{-1})x, (AA^{-1})y) \\ &= \omega(A(A^{-1}x), A(A^{-1}y)) \\ &= \omega(A^{-1}x, A^{-1}y). \end{aligned}$$ [6]

(iv) As we have mentioned, the reasons why G forms a group are the same as (iii).[3] It is symmetric because

$$B_{3,1}(\boldsymbol{u}, \boldsymbol{v}) = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4 = v_1u_1 + v_2u_2 + v_3u_3 - v_4u_4 = B_{3,1}(\boldsymbol{v}, \boldsymbol{u}).$$ [6]

$B_{3,1}$ is not an inner product, because $B_{3,1}(e_4, e_4) = -1 < 0$.[3] □

Problem 2 (Exercise 4.5.3 (13 pts))**.** Show, directly from the definition of matrix exponentiation, that

$$A = \begin{pmatrix} & -\theta \\ \theta & \end{pmatrix}$$

implies

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Solution. We first prove that

$$A^n = \begin{cases} \begin{pmatrix} & (-1)^{\frac{n+1}{2}}\theta^n \\ (-1)^{\frac{n-1}{2}}\theta^n & \end{pmatrix} & \text{if } n \text{ is odd;} \\ \begin{pmatrix} (-1)^{\frac{n}{2}}\theta^n & \\ & (-1)^{\frac{n}{2}}\theta^n \end{pmatrix} & \text{otherwise.} \end{cases}$$ [3]

They are clear for $n = 1$ and $n = 2$. Suppose it is true for n. When n is odd, then

$$A^{n+1} = A^nA = \begin{pmatrix} & (-1)^{\frac{n+1}{2}}\theta^n \\ (-1)^{\frac{n-1}{2}}\theta^n & \end{pmatrix} \begin{pmatrix} & -\theta \\ \theta & \end{pmatrix} = \begin{pmatrix} (-1)^{\frac{n+1}{2}}\theta^{n+1} & \\ & (-1)^{\frac{n+1}{2}}\theta^{n+1} \end{pmatrix},$$

when n is even, then

$$A^{n+1} = A^nA = \begin{pmatrix} (-1)^{\frac{n}{2}}\theta^n & \\ & (-1)^{\frac{n}{2}}\theta^n \end{pmatrix} \begin{pmatrix} & -\theta \\ \theta & \end{pmatrix} = \begin{pmatrix} & (-1)^{\frac{n+2}{2}}\theta^n \\ (-1)^{\frac{n}{2}}\theta^n & \end{pmatrix}.$$

Hence by induction, the conclusion is correct.[5]

Thus, by the definition,

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2} + \cdots + \frac{A^{2n}}{(2n)!} + \frac{A^{2n+1}}{(2n+1)!} + \cdots \\ &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} & -\theta \\ \theta & \end{pmatrix} + \cdots + \begin{pmatrix} (-1)^n\theta^{2n} & \\ & (-1)^n\theta^{2n} \end{pmatrix} + \begin{pmatrix} & (-1)^{n+1}\theta^{2n+1} \\ (-1)^n\theta^{2n+1} & \end{pmatrix} + \cdots \\ &= \begin{pmatrix} \sum_{n=1}^{\infty}(-1)^n\theta^{2n} & \sum_{n=1}^{\infty}(-1)^{n+1}\theta^{2n+1} \\ \sum_{n=1}^{\infty}(-1)^n\theta^{2n+1} & \sum_{n=1}^{\infty}(-1)^n\theta^{2n} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$ [5]

□

Problem 3 (Exercise 4.5.4 (7 pts))**.** Suppose D is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_k$. By computing the powers D^n show that e^D is a diagonal matrix with diagonal entries $e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_k}$.

Solution. We first prove that $D^n = \mathrm{diag}(\lambda_1^n, \lambda_2^n, \cdots, \lambda_k^n)$. First it is clear when $n = 0$. Suppose it is true for n, then

$$D^{n+1} = D^n D = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_k^n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix} = \begin{pmatrix} \lambda_1^{n+1} & & & \\ & \lambda_2^{n+1} & & \\ & & \ddots & \\ & & & \lambda_k^{n+1} \end{pmatrix}.^{[4]}$$

Thus by definition,

$$e^D = I + D + \frac{D^2}{2} + \cdots = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix} + \cdots + \begin{pmatrix} \frac{\lambda_1^n}{n!} & & & \\ & \frac{\lambda_2^n}{n!} & & \\ & & \ddots & \\ & & & \frac{\lambda_k^n}{n!} \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + \lambda_1 + \cdots + \frac{\lambda_1^n}{n!} + \cdots & & & \\ & 1 + \lambda_2 + \cdots + \frac{\lambda_2^n}{n!} + \cdots & & \\ & & \ddots & \\ & & & 1 + \lambda_k + \cdots + \frac{\lambda_k^n}{n!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_k} \end{pmatrix}.^{[3]}$$

□