Math 4500 HW #07 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 4.3.5 (7 pts))**.** Use bilinearity, or otherwise, show that $U, V \in \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$ implies $[U, V] \in \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$.

Solution. Suppose $U = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}, V = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. By definition of quaternion multiplication

$$\begin{aligned}[U, V] &= UV - VU = (a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) \\ &= (-ax + ay\boldsymbol{k} - az\boldsymbol{j} - by + bz\boldsymbol{i} - bx\boldsymbol{k} - cz + cx\boldsymbol{j} - cy\boldsymbol{i}) \\ &- (-ax - ay\boldsymbol{k} + az\boldsymbol{j} - by - bz\boldsymbol{i} + bx\boldsymbol{k} - cz - cx\boldsymbol{j} + cy\boldsymbol{i}) \\ &= 2(bz - cy)\boldsymbol{i} + 2(cx - az)\boldsymbol{j} + 2(ay - bx)\boldsymbol{k} \in \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}.\end{aligned}$$

□

Problem 2 (Exercise 4.4.1 (3 pts))**.** Prove the Jacobi identity by using the definition $[X, Y] = XY - YX$ to expand $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]]$.

Solution. This verification comes from the definition

$$\begin{aligned}[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] &= [X, YZ - ZY] + [Z, XY - YX] + [Y, ZX - XZ] \\ &= XYZ - XZY - YZX + ZYX \\ &+ ZXY - ZYX - XYZ + YXZ \\ &+ YZX - YXZ - ZXY + XZY \\ &= 0.\end{aligned}$$

□

Problem 3 (Exercise 5.2.8 (15 pts))**.** Deduce from Exercise 5.2.6 and 5.2.7 that each matrix in $SO(3)$ equals e^X for some skew-symmetric X.

Solution. First we compute e^B for $B = \begin{pmatrix} & -\theta & \\ \theta & & \\ & & \end{pmatrix}$. Denote $P = \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$, then we know that

$$e^B = e^{\theta P} = \sum_{n=0}^{\infty} (\theta P)^n = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}.$$

Suppose A is an orthogonal matrix, then

$$\begin{aligned}Ae^B A^T &= A \left(\sum_{n=0}^{\infty} B^n \right) A^T \\ &= \sum_{n=0}^{\infty} AB^n A^T \\ &= \sum_{n=0}^{\infty} (ABA^T)^n \\ &= e^{ABA^T}.\end{aligned}$$

We know that for any orthogonal matrix $A \in O(n)$, we have a decomposition

$$A = CHC^T$$

where $C \in O(n)$, H is a block-diagonal matrix having the form $\operatorname{diag}\{R_1, \cdots, R_m, 1, \cdots, 1\}$ and $R_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$ for some real θ_i. Here in $\mathbb{R}^3$

$$A = C \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} C^T = Ce^B C^T = e^{CBC^T},$$

where $C \in O(3)$ and thus $CBC^T \in SO(3)$. □

Problem 4 (Exercise 5.3.6 (13 pts))**.** Show that the skew-Hermitian matrices in the tangent space of $SU(2)$ can be written in the form $b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ where $b, c, d \in \mathbb{R}$ and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are matrices with the same multiplication table as the quaternions $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Solution. Notice that all the matrices in the tangent space have the form

$$A = \begin{pmatrix} di & b + ci \\ -b + ci & -di \end{pmatrix},$$

since $A + \bar{A}^T = 0$ and $\operatorname{Trace}(A) = 0$. First we compute the linear combination for specific $\boldsymbol{i}_0 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \boldsymbol{j}_0 = \begin{pmatrix} & -i \\ -i & \end{pmatrix}, \boldsymbol{k}_0 = \begin{pmatrix} i & \\ & -i \end{pmatrix}$. It is obviously that

$$\begin{aligned} A &= \begin{pmatrix} di & b + ci \\ -b - ci & -di \end{pmatrix} \\ &= -b \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} - c \begin{pmatrix} & -i \\ -i & \end{pmatrix} + d \begin{pmatrix} i & \\ & -i \end{pmatrix} \\ &= -b\boldsymbol{i}_0 - c\boldsymbol{j}_0 + d\boldsymbol{k}_0. \end{aligned}$$

Then for arbitrary matrices with the same multiplication table as the quaternions $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we have some matrix $C \in SO(3)$ s.t. $C[\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}] = [\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0]$, i.e. we have some change of basis s.t. the multiplication table of bases is preserved. Hence

$$\begin{aligned} A &= -b\boldsymbol{i}_0 - c\boldsymbol{j}_0 + d\boldsymbol{k}_0 \\ &= -b'\boldsymbol{i} - c'\boldsymbol{j} + d'\boldsymbol{k} \end{aligned}$$

where $\begin{pmatrix} b' \\ c' \\ d' \end{pmatrix} = C \begin{pmatrix} b \\ c \\ d \end{pmatrix}$. □

Problem 5 (Exercise 5.3.7 (10 pts))**.** Also find the tangent space of $Sp(1)$.

Solution. Suppose $q(t)$ be a smooth path of $Sp(1)$ originating at I, then

$$q(t)\overline{q(t)} = I.$$

Take the derivative, then

$$q'(t)\overline{q(t)} + q(t)\overline{q'(t)} = 0.$$

Since $q(t) = I$, for $t = 0$ the equation becomes

$$q'(0) + \overline{q'(0)} = 0,$$

hence the tangent vector should be a pure imaginary quaternion. Conversely, for any pure imaginary quaternion $p = b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$, set

$$q(t) := e^{tp} \in Sp(1)$$

where $t \in [-1, 1]$, then apparently $q'(0) = p = b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$. □

Problem 6 (Exercise 5.3.8 (7 pts))**.** Prove that $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$.

Solution. Suppose that $X = (X_{i,j})_{i,j=1,\cdots,n}$ and $Y = (Y_{i,j})_{i,j=1,\cdots,n}$, then

$$\begin{aligned}\mathrm{Trace}(XY) &= \mathrm{Trace}\left(\left(\sum_{k=1}^{n} X_{i,k}Y_{k,j}\right)_{i,j=1,\cdots,n}\right)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} X_{i,k}Y_{k,i}\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Trace}(YX) &= \mathrm{Trace}\left(\left(\sum_{k=1}^{n} Y_{i,k}X_{k,j}\right)_{i,j=1,\cdots,n}\right)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} Y_{i,k}X_{k,i}.\end{aligned}$$

□