Math 4500 HW #09 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (7+15+3=25 pts)**.** Recall the definition of a homomorphism of Lie algebras. We say two abstract Lie algebras are isomorphic if there is a bijective homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$.

1. Prove that $\mathfrak{sl}_2(\mathbb{C}) := \{A \in M_2(\mathbb{C}) \mid \operatorname{Trace} A = 0\}$ is a linear Lie algebra with the bracket $[A, B] = AB - BA$. Also prove that $\mathfrak{sl}_2(\mathbb{C})$ admits a $\mathbb{C}$-linear basis $X = \begin{pmatrix} & 1 \\ & \end{pmatrix}$, $Y = \begin{pmatrix} & \\ 1 & \end{pmatrix}$ and $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ with $[X, Y] = H$, $[H, X] = 2X$, and $[H, Y] = -2Y$.
2. Bearing in mind that for any vector space V, $\mathfrak{gl}(V) := \{A \in \operatorname{End}(V)\}$ with the bracket $[A, B] = AB - BA$ is an abstract Lie algebra, prove that we have a Lie algebra homomorphism
$$\begin{aligned} \operatorname{ad} : \mathfrak{sl}_2(\mathbb{C}) &\to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})) \\ x &\mapsto [x, -] \end{aligned}$$
which is injective but not surjective.
3. Find the eigenvalues of the linear map $\operatorname{ad}(H) : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$.

Solution. (i) If matrix $A \in \mathfrak{sl}_2(\mathbb{C})$, then $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Thus

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} + b \begin{pmatrix} & \\ 1 & \end{pmatrix} + c \begin{pmatrix} & 1 \\ & \end{pmatrix} = aH + bY + cX,$$

and clearly the expression is unique. Hence X, Y, H form a basis.

By direct computations,

$$[X, Y] = \begin{pmatrix} & 1 \\ & \end{pmatrix} \begin{pmatrix} & \\ 1 & \end{pmatrix} - \begin{pmatrix} & \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ & \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

$$[H, X] = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & 1 \\ & \end{pmatrix} - \begin{pmatrix} & 1 \\ & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = 2 \begin{pmatrix} & 1 \\ & \end{pmatrix},$$

and

$$[H, Y] = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & \\ 1 & \end{pmatrix} - \begin{pmatrix} & \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = -2 \begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

(ii) Since the Lie bracket of $\mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C}))$ is bilinear, ad is a linear map. For any $z \in \mathfrak{sl}_2(\mathbb{C})$

$$\begin{aligned} \operatorname{ad}([x, y])(z) &= [[x, y], z] \\ &= -[[z, x], y] - [[y, z], x] \\ &= \operatorname{ad}(y)(-\operatorname{ad}(x)(z)) + \operatorname{ad}(x)(\operatorname{ad}(y)(z)) \\ &= (\operatorname{ad}(x) \circ \operatorname{ad}(y) - \operatorname{ad}(y) \circ \operatorname{ad}(x))(z) \\ &= [\operatorname{ad}(x), \operatorname{ad}(y)](z), \end{aligned}$$ [7]

where the second equation comes from Jacobi identity. Hence ad is a homomorphism. ad is apparently not surjective since $\mathfrak{sl}_2(\mathbb{C})$ is of dimension 3 but $\mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C}))$ is of dimension 9.[3] Finally to see ad is injective, it suffices to prove that $\mathrm{ad}(X), \mathrm{ad}(Y), \mathrm{ad}(H)$ are linearly independent. Suppose we have a linear combination

$$a\mathrm{ad}(X) + b\mathrm{ad}(Y) + c\mathrm{ad}(H) = 0,$$

then for any $A \in \mathfrak{sl}_2(\mathbb{C})$

$$a\mathrm{ad}(X)(A) + b\mathrm{ad}(Y)(A) + c\mathrm{ad}(H)(A) = 0.$$

Take $A = X$ and $A = H$ respectively, and by the linearly independence of X, Y, H, we have $a = b = c = 0$, which means $\mathrm{ad}(X), \mathrm{ad}(Y), \mathrm{ad}(H)$ are linearly independent.

(iii) By part (i), $\mathrm{ad}(H)$ has eigenvectors X, Y, H and eigenvalues $2, -2, 0$ respectively. □

Problem 2 (Exercise 6.5.4 (10 pts))**.** Prove that each 4×4 skew-symmetric matrix is uniquely decomposable as a sum

$$\begin{pmatrix} & -a & -b & -c \\ a & & -c & b \\ b & c & & -a \\ c & -b & a & \end{pmatrix} + \begin{pmatrix} & -x & -y & -z \\ x & & z & -y \\ y & -z & & x \\ z & y & -x & \end{pmatrix}.$$

Solution. For any 4×4 skew-symmetric matrix $\begin{pmatrix} & -\alpha & -\beta & -\gamma \\ \alpha & & -\delta & -\epsilon \\ \beta & \delta & & -\eta \\ \gamma & \epsilon & \eta & \end{pmatrix}$, we have

$$\begin{pmatrix} & -\alpha & -\beta & -\gamma \\ \alpha & & -\delta & -\epsilon \\ \beta & \delta & & -\eta \\ \gamma & \epsilon & \eta & \end{pmatrix} = \begin{pmatrix} & -\frac{\alpha+\eta}{2} & -\frac{\beta-\epsilon}{2} & -\frac{\gamma+\delta}{2} \\ \frac{\alpha+\eta}{2} & & -\frac{\gamma+\delta}{2} & \frac{\beta-\epsilon}{2} \\ \frac{\beta-\epsilon}{2} & \frac{\gamma+\delta}{2} & & -\frac{\alpha+\eta}{2} \\ \frac{\gamma+\delta}{2} & -\frac{\beta-\epsilon}{2} & \frac{\alpha+\eta}{2} & \end{pmatrix} + \begin{pmatrix} & -\frac{\alpha-\eta}{2} & -\frac{\beta+\epsilon}{2} & -\frac{\gamma-\delta}{2} \\ \frac{\alpha-\eta}{2} & & \frac{\gamma-\delta}{2} & -\frac{\beta+\epsilon}{2} \\ \frac{\beta+\epsilon}{2} & -\frac{\gamma-\delta}{2} & & \frac{\alpha-\eta}{2} \\ \frac{\gamma-\delta}{2} & \frac{\beta+\epsilon}{2} & -\frac{\alpha-\eta}{2} & \end{pmatrix}. \quad (1)$$

The uniqueness comes from the fact that writing into the form of a sum gives us a linear system of equations. □

Problem 3 (Exercise 6.5.6 (15 pts))**.** Deduce from Exercises 6.5.4 and 6.5.5 that $\mathfrak{so}(4)$ is isomorphic to the direct product $\mathfrak{so}(3) \times \mathfrak{so}(3)$ (also known as the direct sum and commonly written $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$).

Solution. Let $I = -\boldsymbol{E}_{12} - \boldsymbol{E}_{34}, J = -\boldsymbol{E}_{13} + \boldsymbol{E}_{24}, K = -\boldsymbol{E}_{14} - \boldsymbol{E}_{23}$, and $L = -\boldsymbol{E}_{12} + \boldsymbol{E}_{34}, M = -\boldsymbol{E}_{13} - \boldsymbol{E}_{24}, N = -\boldsymbol{E}_{14} + \boldsymbol{E}_{23},$, we have that

$$\begin{pmatrix} & -\alpha & -\beta & -\gamma \\ \alpha & & -\delta & -\epsilon \\ \beta & \delta & & -\eta \\ \gamma & \epsilon & \eta & \end{pmatrix} = \frac{\alpha+\eta}{2}I + \frac{\beta-\epsilon}{2}J + \frac{\gamma+\delta}{2}K + \frac{\alpha-\eta}{2}L + \frac{\beta+\epsilon}{2}M + \frac{\gamma-\delta}{2}N.$$

And also by Exercise 6.5.5. and direct computation, $[I, J] = 2K, [J, K] = 2I, [K, I] = 2J$ and $[L, M] = 2N, [M, N] = 2L, [N, L] = 2M$.[10] But also we have that $[I, L] = [I, M] = [I, N] = [J, L] = [J, M] = [J, N] = [K, L] = [K, M] = [K, N] = 0$, hence we have two copies of $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$, and by the linear independence we know that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$. □

Problem 4 (Exercise 8.1.3 (5 pts))**.** Give an example of an infinite intersection of open sets that is not open.

Solution. Let $X = \mathbb{R}$ be the topological space with the Euclidean topology, and let $U_n := (-\frac{1}{n}, \frac{1}{n})$ be the sequence of open sets. Then

$$\bigcap_{n=1}^{\infty} U_n = \{0\},$$

which is a closed subset instead of an open set. □

Problem 5 (Exercise 8.2.1 (15 pts))**.** Prove that $U(n)$, $SU(n)$ and $Sp(n)$ are closed subsets of the appropriate matrix spaces.

Solution. Define the map

$$\begin{aligned} \varphi_1 : M_n(\mathbb{C}) &\to M_n(\mathbb{C}) \\ A &\mapsto A\bar{A}^T, \end{aligned}$$

then φ_1 is obviously continuous and $U(n)$ is the preimage of the point $\{I\}$, hence it is closed in $M_n(\mathbb{C})$. Similarly, define

$$\begin{aligned} \varphi_2 : U(n) &\to \mathbb{C} \\ A &\mapsto \det A \end{aligned}$$

and

$$\begin{aligned} \varphi_3 : M_n(\mathbb{H}) &\to M_n(\mathbb{H}) \\ A &\mapsto A\bar{A}^T, \end{aligned}$$

where $SU(n) = \varphi_2^{-1}(1)$ and $Sp(n) = \varphi_3^{-1}(I)$, i.e. $SU(n)$ and $Sp(n)$ are preimages of two closed points. In conclusion, $U(n)$ is closed in $M_n(\mathbb{C})$, $SU(n)$ is closed in $U(n)$ hence in $M_n(\mathbb{C})$, and $Sp(n)$ is closed in $M_n(\mathbb{H})$. □