Math 4500 HW #10 Solutions

Instructor: Birgit Speh
TA: Guanyu Li

This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 8.2.2 (7 pts))**.** Show that $GL_n(\mathbb{C})$ is an open subset of $M_n(\mathbb{C})$.

Solution. Considering the continuous function

$$\det : M_n(\mathbb{C}) \to \mathbb{C},$$

$GL_n(\mathbb{C})$ is the preimage of $\mathbb{C} - \{0\}$. The continuity tells us that $GL_n(\mathbb{C})$ is open. □

Problem 2 (Exercise 8.3.4 (7 pts))**.** Give an example of a continuous function f on $\mathbb{R}$ and a set S s.t.

$$f(\bar{S}) \neq \overline{f(S)},$$

where $\bar{S}$ is the closure of S.

Solution. Consider

$$f(x) = \arctan x,$$

and $S = \mathbb{R}$, then $(-\frac{\pi}{2}, \frac{\pi}{2}) = f(S) = f(\bar{S}) \subsetneq [-\frac{\pi}{2}, \frac{\pi}{2}] = \overline{f(S)}$. □

Problem 3 (Exercise 8.4.5 (10 pts))**.** Show that $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are not compact.

Solution. First it is not hard to prove that a continuous image of a compact set is still compact (you could prove it yourself). Hence we consider the continuous map $\det : GL(n, \mathbb{C}) \to \mathbb{C}$, its image is $\mathbb{C} - \{0\}$ which is not compact, hence $GL(n, \mathbb{C})$ is not compact.

For the second part, we shall use the fact that a subset C is compact in an Euclidean space if and only if it is bounded and closed. But the matrices

$$A_k := \begin{pmatrix} k & & & & \\ & \frac{1}{k} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

are element in $SL_n(\mathbb{C})$ however $\|A_k\| > k \to +\infty$ as $k \to +\infty$. Hence $SL_n(\mathbb{C})$ is not bounded. □

Problem 4 (Exercise 8.6.1 (6 pts))**.** Write $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ as the product of two matrices in $SL(2, \mathbb{C})$ with entries $0, i$ or $-i$.

Solution.

$$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} = \begin{pmatrix} i & i \\ & -i \end{pmatrix} \begin{pmatrix} i & \\ & -i \end{pmatrix}.$$

Notice that two matrices are traceless. □

Problem 5 (Exercise 8.6.2 (10 pts))**.** Deduce from Exercise 8.6.1 and Exercise 5.6.4 that $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} = e^{X_1} e^{X_2}$ for some $X_1, X_2 \in T_I(SL(2, \mathbb{C}))$.

Solution. Suppose that $\begin{pmatrix} i & i \\ & -i \end{pmatrix} = e^{X_1}$. By Caylay-Hamilton, we can use the result from Problem 5.6.4.,

$$e^{X_1} = \cos(\sqrt{\det(X_1)})I + \frac{\sin(\sqrt{\det(X_1)})}{\sqrt{\det(X_1)}} X_1.$$

This identity is actually gives us a system of linear equations. Thus we have that

$$X_1 = \frac{\pi}{2} \begin{pmatrix} i & i \\ & -i \end{pmatrix}.$$

Similarly we have $X_2 = \frac{\pi}{2} \begin{pmatrix} i & \\ & -i \end{pmatrix}$. □

Problem 6 (Exercise 8.1.2 (0 pts)).

Solution. □

Problem 7 (Exercise 8.3.1 (0 pts)).

Solution. □

Problem 8 (Exercise 8.3.2 (0 pts)).

Solution. □

Problem 9 (Exercise 8.5.1 (0 pts)).

Solution. □

Problem 10 (Exercise 8.5.2 (0 pts)).

Solution. □