Math 4500 HW #11 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 7.2.1 (10 pts))**.** Deduce from exponentiation of tangent vectors that

$$T_I(G) = \{X \mid e^{tX} \text{ for all } t \in \mathbb{R}\}.$$

Solution. For each $X \in T_I(G)$ and $t \in \mathbb{R}$, we know that $tX \in T_I(G)$. By the proof of the textbook, we know that e^{tX} is an element in G. Therefore

$$T_I(G) \subseteq \{X \mid e^{tX} \text{ for all } t \in \mathbb{R}\}.$$

Conversely, for each $X \in \{X \mid e^{tX} \text{ for all } t \in \mathbb{R}\}$, define $\alpha(t) = e^{tX}$, hence $\alpha(t)$ is a continuous path on G s.t. $\alpha(0) = I$. Take the derivative then $\alpha'(0) = X$. Hence

$$T_I(G) \supseteq \{X \mid e^{tX} \text{ for all } t \in \mathbb{R}\}.$$

□

Problem 2 (Exercise 7.2.2 (5 pts))**.** Given X as the tangent vector to e^{tX}, and Y as the tangent vector to e^{tY}, show that $X+Y$ is the tangent vector to $A(t) = e^{tX}e^{tY}$.

Solution. Take $A(t)$ as defined, then

$$\begin{aligned} A'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(e^{tX}e^{tY}) \\ &= Xe^{tX}e^{tY} + e^{tX}Ye^{tY}, \end{aligned}$$

hence $A'(0) = X + Y$. □

Problem 3 (Exercise 7.2.3 (5 pts))**.** Similarly, show that if X is a tangent vector then so is rX for any $r \in \mathbb{R}$.

Solution. Similar to last problem, let $B(t) = e^{rtX}$, then

$$B'(t) = rXe^{rtX}$$

hence $B'(0) = rX$. □

Problem 4 (Exercise 7.2.4 (10 pts))**.** Suppose that, for each A in some neighborhood N of I in G, there is a smooth function $A(t)$, with values in G, such that $A(\frac{1}{n}) = A^{\frac{1}{n}}$ for $n \in \mathbb{N}^*$. Show that $A'(0) = \log A$, so that $\log A \in T_I(G)$.

Solution. Suppose we have a path $A(t)$ s.t. $A(0) = I$ with $A(\frac{1}{n}) = A^{\frac{1}{n}}$ for $n \in \mathbb{N}^*$, then

$$A'(0) = \lim_{n\to+\infty} \frac{A(\frac{1}{n} - I)}{\frac{1}{n}} = \lim_{n\to+\infty} n \log A(\frac{1}{n}) = \lim_{n\to+\infty} n \log A^{\frac{1}{n}}.$$

Since that A is commutative with itself, $n \log A^{\frac{1}{n}} = \log A^{\frac{1}{n}} + \cdots + \log A^{\frac{1}{n}} = \log(A^{\frac{1}{n}} \cdots A^{\frac{1}{n}}) = \log A$. (Here is an issue of well-definedness. Instead of working for it, it is OK to just take that $\log A^{\frac{k}{n}}$ is well-defined for all $1 \leq k \leq n$.) □

Problem 5 (Exercise 7.2.5 (10 pts))**.** Suppose, conversely, that log maps some neighborhood N of I in G into $T_I(G)$. Explain why we can assume that N is mapped by log onto an ϵ-ball $N_\epsilon(0)$ in $T_I(G)$.

Solution. It suffices to find some $\epsilon > 0$ s.t. for any $|X| < \epsilon$, $|e^X - I| < 1$, so that by the textbook for any $X \in N_\epsilon(0)$,

$$\log e^X = X \in N_\epsilon(0).$$

Notice that

$$\begin{aligned} |e^X - I| &= \left| \sum_{n=1}^{\infty} \frac{X^n}{n!} \right| \\ &\leq \sum_{n=1}^{\infty} \frac{|X^n|}{n!} \\ &\leq \sum_{n=1}^{\infty} \frac{|X|^n}{n!} \\ &\leq \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} = e^\epsilon - 1 \end{aligned}$$

where the first inequality comes from the continuity of the norm (NOT JUST THE TRIANGLE INEQUALITY). Hence we can just take one $\epsilon < \ln 2$. □

Problem 6 (Exercise 7.2.6 (5 pts))**.** Take N as in Exercise 7.2.4., and $A \in N$, show that $t \log A \in T_I(G)$ for all $t \in [0, 1]$, and deduce that $A^{\frac{1}{n}}$ exists for $n \in \mathbb{N}^*$.

Solution. Directly from 7.2.3.. □