Math 4500 HW #12 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 7.6.3 (10 pts))**.** Suppose that $e^X e^Y = e^Y e^X$. Show that $XY = YX$.

Solution. Suppose that $X = \log(I + (e^X - I))$ and $Y = \log(I + (e^Y - I))$, then we know that

$$\begin{aligned} &= \sum_{n=1}^{+\infty}(-1)^{n-1}(e^X - I)^n \sum_{n=1}^{+\infty}(-1)^{n-1}(e^Y - I)^n \\ &= \sum_{n,m=1}^{+\infty}(-1)^{n+m}(e^X - I)^n(e^Y - I)^m. \end{aligned}$$

Since that $e^X e^Y = e^Y e^X$, we have $(e^X - I)^n(e^Y - I)^m = (e^Y - I)^m(e^X - I)^n$, thus

$$\begin{aligned} \sum_{n,m=1}^{+\infty}(-1)^{n+m}(e^X - I)^n(e^Y - I)^m &= \sum_{n,m=1}^{+\infty}(-1)^{n+m}(e^Y - I)^m(e^X - I)^n \\ &= \log(I + (e^Y - I))\log(I + (e^X - I)) = YX. \end{aligned}$$

□

Problem 2 (Exercise 7.6.4 (8 pts))**.** Deduce from Exercise 7.6.3. that $e^X e^Y = e^{X+Y}$ if and only if $XY = YX$.

Solution. Since $X + Y = Y + X$, if $e^X e^Y = e^{X+Y}$ then

$$e^X e^Y = e^{X+Y} = e^{Y+X} = e^Y e^X,$$

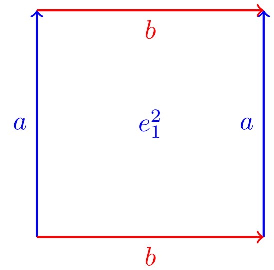
and by previous problem, $XY = YX$. Converse proposition was proved in the text. □

Problem 3 (Exercise 9.1.1 (7 pts))**.** Find algebraic properties showing that the groups $O(2)$, $SO(2)$, and $\mathbb{R}$ are not isomorphic.

Solution. Notice that $O(2)$ is not abelian and the other two are abelian. There are elements in $SO(2)$ with finite orders, i.e. $\mathbb{Z}/n\mathbb{Z}$ is a subgroup of $SO(2)$, but for any $0 \neq r \in \mathbb{R}$, if $nr = 0$ for some integer n, then $n = 0$. □

Problem 4 (Exercise 9.1.2 (5 pts))**.** Explain why it is appropriate to call $S^1 \times S^1$, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ the torus, cylinder, and plane respectively.

Solution. $S^1 \times S^1$ is gluing opposite sides of a square, say



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If we find a representative of it in $\mathbb{R}^3$, the gluing process gives us a torus. Similar to $S^1 \times \mathbb{R}$, this is the gluing of a infinitely long stripe, which gives us a(n) (infinitely long) cylinder. □

Problem 5 (Exercise 9.1.3 (10 pts))**.** Show that the three groups have the same Lie algebra. Describe its underlying vector space and Lie bracket operation.

Solution. By definition, the Lie algebra of a group is the set of the tangent vectors of all possible smooth paths going through the identity. But the tangent vector is a local notion, it suffices to prove that these three groups have the same local property at the identity. Notice that $S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$ and $S^1 \times \mathbb{R} \cong \mathbb{R}^2/\mathbb{Z}$, so there are open neighborhoods of the identities of the three groups s.t. they are homeomorphic, hence they have the same Lie algebra.

If we just compute the Lie algebra of $\mathbb{R} \times \mathbb{R}$, the underlying space is $\mathbb{R} \times \mathbb{R}$, with the trivial bracket. □

Problem 6 (Exercise 9.1.4 (15 pts))**.** Distinguish the three groups algebraically and topologically.

Solution. Algebraically, $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is a subgroup of $S^1 \times S^1$, but not a subgroup of $S^1 \times \mathbb{R}$ nor $\mathbb{R} \times \mathbb{R}$. We only need to prove that $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is not a subgroup of $S^1 \times \mathbb{R}$. Suppose not, then we have some injection $\varphi : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to S^1 \times \mathbb{R}$, then since all elements in $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ are of finite order, so their image under φ. Thus $\operatorname{Im} \varphi \subseteq S^1 \times \{0\}$. But $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is not cyclic but all finite subgroup of S^1 is cyclic, a contradiction. Similarly, $\mathbb{Z}/n\mathbb{Z}$ is a subgroup of $S^1 \times \mathbb{R}$ but not of $\mathbb{R} \times \mathbb{R}$.

Topologically, $S^1 \times S^1$ is compact but not simply connected, while $S^1 \times \mathbb{R}$ is noncompact but not simply connected. $\mathbb{R} \times \mathbb{R}$ is noncompact but simply connected. □