Combinatorics - A Toric Algebraic Geometry Approach

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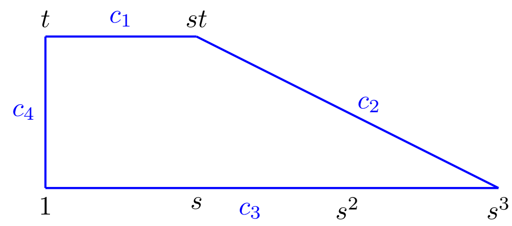
Sum, Product and Power

Contents

Caution. Within this notes we will only work on varieties $/\mathbb{C}$ [1]. You'd better skip all the proofs if this is the first time you learn toric varieties.

Example 1. Given the polytope



,

we then can produce a morphism

$$\begin{aligned}\varphi : \mathbb{A}^2_{(s,t)} &\to \mathbb{P}^5\\ (s,t) &\mapsto [1, s, s^2, s^3, t, st].\end{aligned}$$

Let $X := \overline{\varphi(\mathbb{A}^2)}$, then X is our first toric variety.

1 A Bad Definition of Toric Varieties

Definition. The n-torus is defined to be

$$(\mathbb{C}^*)^n := \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}],$$

where the group structure is given component-wisely.

Proposition 1 (Good properties)**.** *1. Let T_1 and T_2 be tori and let $\varphi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then the image of φ is closed in T_2.*

2. Let T be a torus and let $H \subseteq T$ be an irreducible subvariety of T that is a subgroup. Then H is a torus.

Definition. A *character* of torus $(\mathbb{C}^*)^n$ is a morphism $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$. Dually, a *cocharacter*, or say a *one parameter subgroup* is a morphism $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$.

Example 2. Given an $m = (a_1, \cdots, a_n) \in \mathbb{Z}^n$, there is a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^m(t_1, \cdots, t_n) = t_1^{a_1} \cdots t_n^{a_n}.$$

The amazing thing is that all characters come in this way. Also, there is a 1-PS associated with m

$$\begin{aligned}\lambda^m : \mathbb{C}^* &\to (\mathbb{C}^*)^n\\ t &\mapsto (t_1^{a_1}, \cdots, t_n^{a_n}).\end{aligned}$$

All 1-PS's come in this way as well.

Assume that a torus T acts linearly on a finitely dimensional vector space W over $\mathbb{C}$. A basic result is that the maps $w \mapsto t \cdot w$ are simultaneously diagonalizable as follows. Given $m \in M$, define the eigenvector space

$$W_m := \{w \in W \mid t \cdot w = \chi^m(t) w \text{ for all } t \in T\}.$$

Then one can show that $W \cong \bigoplus_{m \in \mathbb{Z}^n} W_m$.

Definition (Bad Definition)**.** A *toric variety* is an irreducible variety V containing a torus $T := (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on V.

Example 3. Consider the canonical embedding $\mathbb{C}^* \hookrightarrow \mathbb{C}$ given by $\mathbb{C}[x] \to \mathbb{C}[x, x^{-1}]$. Generalizing this gives that all affine spaces are toric varieties.

Definition. A lattice is a free abelian group of finite rank. Thus a lattice of rank n is isomorphic to $\mathbb{Z}^n$.

By Example 2, we say the torus $(\mathbb{C}^*)^n$ has lattice of characters $M = \mathbb{Z}^n$ and has lattice of 1-PS $N = M^\vee \cong \mathbb{Z}^n$.

Theorem 2. *Given a torus $(\mathbb{C}^*)^n$, and a set $A = \{m_1, \cdots, m_s\} \subseteq M$, then there is a map*

$$\Phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^s,$$

defined by

$$\Phi_A(t) = (\chi^{m_1}(t), \cdots, \chi^{m_s}(t)).$$

Let X_A be the Zariski closure of the image of $\Phi_A(t)$. Then X_A is an toric variety of character lattice $\mathbb{Z}A$.

Theorem 2 is much more concrete than our definition. However it is still hard to see the Zariski closure. We will later give another definition and prove that they are equivalent.

Example 4. Consider $\sigma^\vee \cap M$ be the rational cone generated by vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$, since

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix} = 0,$$

the affine coordinate ring is $\mathbb{C}[x, y, z, w]/(xy - zw)$. Since $V = \operatorname{Spec} \mathbb{C}[x, y, z, w]/(xy - zw)$ is in $\mathbb{C}^4$,

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1t_2t_3^{-1}) \mid t_i \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^3,$$

where the isomorphism is $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1t_2t_3^{-1})$.

2 Cones and Fourier-Motzkin

Definition. Let V be a finite dimensional vector space over $\mathbb{R}$, $S \subseteq V$ is a non-empty subset.

1. S be a *(convex) cone* if $\forall x, y \in S, \alpha, \beta \in \mathbb{R}$, if $\alpha, \beta \geq 0$ then $\alpha x + \beta y \in S$.

2. S be a *(convex) set* if $\forall x, y \in S, \alpha, \beta \in \mathbb{R}$, if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ then $\alpha x + \beta y \in S$.

3. A convex polyhedral cone is of the form

$$\sigma = \operatorname{cone}(S) = \left\{ \sum_{u \in S} c_u u \mid c_u \geq 0 \right\},$$

   where $S \subseteq V$ is finite. We say that σ is generated by S.

Example 5 (Key examples).

1. Let $A \in \mathbb{R}^{m \times n}$, $A = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n]$ where $\boldsymbol{a}_i \in \mathbb{R}^m$. Define

$$\operatorname{vcone}(A) := \{x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n \mid x_i \geq 0\} = \{A\boldsymbol{x} \mid \boldsymbol{x} \geq \boldsymbol{0}\},$$

   where a cone of this form is called finitely generated.

2. Notations as before, define

$$\operatorname{conv}(A) = \{x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\},$$

   where a set of this form is called a polytope.

3. Let $B = \begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix}$ where $\boldsymbol{b}_i \in (\mathbb{R}^m)^*$. Define

$$\mathrm{hcone}(B) := \{\boldsymbol{y} \in (\mathbb{R}^m)^* \mid \langle \boldsymbol{y}, \boldsymbol{b}_i \rangle \leq 0\} = \{\boldsymbol{y} \in (\mathbb{R}^m)^* \mid \ B\boldsymbol{y} \leq 0\}.$$

4. $A \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m$, define

$$P(A, \boldsymbol{b}) := \{\boldsymbol{x} \in \mathbb{R}^n \mid \ A\boldsymbol{x} \leq \boldsymbol{b}\}.$$

   This is called to be polyhedral.

Definition. Note that the span of a cone σ is the smallest subspace of V containing σ. Then the relative interior of σ, denoted by $\mathrm{relint}\ \sigma$, is the interior of σ in its span.

Lemma 1. *For a cone σ,*

$$u \in \mathrm{relint}\ \sigma \Leftrightarrow \langle m, u \rangle > 0$$

Definition. For a cone $\sigma \subseteq V$, its dual cone is

$$\sigma^\vee := \{\boldsymbol{y} \in V^* \mid \langle \boldsymbol{y}, \boldsymbol{x} \rangle \leq 0, \forall \boldsymbol{x} \in V\}.$$

Lemma 2. *1. If $\sigma = \mathrm{vcone}(A)$, then $\sigma^\vee = \mathrm{hcone}(A^T)$.*

2. σ is a vcone if and only if it is an hcone.

3. $\sigma^{\vee\vee} = \sigma$.

Theorem 3 (Fourier-Motzkin Elimination)**.**

Theorem 4 (Weyl-Minkowski)**.** *Let $\sigma \subseteq \mathbb{R}^n$ be a cone. Then σ is finitely generated if and only if σ is polyhedral.*

3 Polytope Geometry

Definition. Let $\sigma \subseteq \mathbb{R}^n$ be a polyhedral cone. The *linear space* of σ is the largest subspace contained in σ. The cone σ is said to be *pointed* or *strongly convex* if its linear space is 0.

Proposition 5. *1. The following are equivalent:*

   (a) $\sigma \cap -\sigma = 0$.
   (b) σ is pointed.
   (c) There is a $u \in \sigma^\vee$ with $\sigma \cap U^\perp = 0$.
   (d) $\sigma^\vee$ spans V.
   (e) $\dim \sigma = \dim \sigma^\vee = \dim V$.

2. Any cone $\sigma \subseteq V$ can be written as the sum of a linear space and a pointed cone. In face

$$\sigma = L + \tau,$$

   where $L := \sigma \cap -\sigma$ and $\tau := \sigma \cap L^\perp$ is pointed.

Definition. A *face* of a polyhedral cone $\sigma \subseteq \mathbb{R}^n$ is a subset $\tau \subseteq \sigma$ of the form

$$\tau := \sigma \cap u^\perp$$

for some $u \in \sigma^\vee$. A 1-dimensional face is called an *edge* or an *extremal ray*. A 1-codimensional face is called a facet.

Definition. Given any $u \in \mathbb{R}^n$, we define

$$H_m := \{f \in (\mathbb{R}^n)^* \mid \langle f, u\rangle = 0\}$$

and

$$H_m^+ := \{f \in (\mathbb{R}^n)^* \mid \langle f, u\rangle \geq 0\}.$$

Lemma 3. 1. *σ it self is a face.*

2. *The smallest face is $\sigma \cap -\sigma$.*

3. *A face τ of σ is also a polyhedral cone.*

4. *A face of a face is also a face.*

5. *If $\sigma = \mathrm{vcone}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$, $\boldsymbol{u} \in \sigma^\vee$, then $\tau = \mathrm{vcone}(\boldsymbol{a}_i \mid \langle \boldsymbol{u}, \boldsymbol{a}_i\rangle = 0)$.*

Definition. A cone σ in $V = \mathbb{R}^n$ is said to be *rational* if it is generated by vectors (i.e. vcone) in $\mathbb{Q}^n$ (or equivalently $\mathbb{Z}^n$).

Lemma 4. *A cone $\sigma \subseteq \mathbb{R}^n$ is rational if and only if $\sigma^\vee$ is rational.*

Definition. Let M be a lattice, then $M \cong \mathbb{Z}^n$ for some integer n which is the rank. We have $M \subseteq M_{\mathbb{Q}} := \mathbb{Q}^n \subseteq M_{\mathbb{R}} := \mathbb{R}^n$, and $M_k := M \otimes_{\mathbb{Z}} k$. Let N be the dual lattice $:= \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = M^*$. We say σ is a cone in N if σ is a rational polyhedral cone in $N_{\mathbb{R}}$.

Definition. Let σ be a cone in N, we define

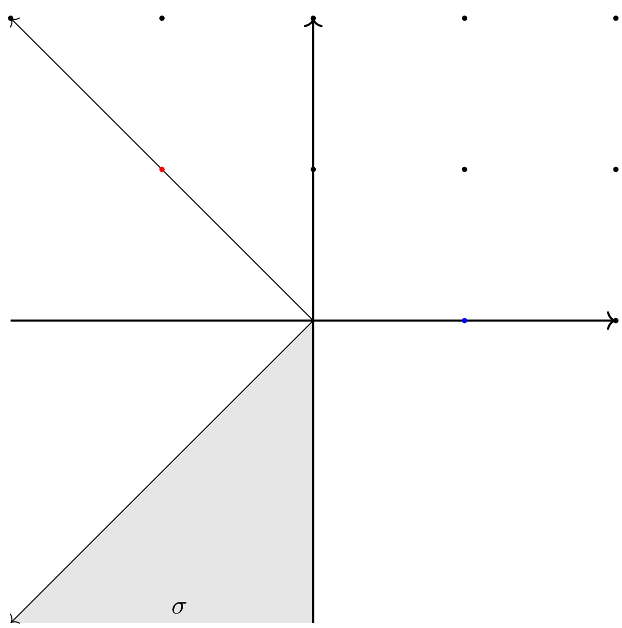
$$S_\sigma := \sigma^\vee \cap M.$$

Note that S_σ is a semi-group. Natural question: why not $S_\sigma := \sigma \cap N$?

Definition. A *semi-group* is a set S with an associative binary operation and an element of identity. We say a semi-group S is *affine* if S is further required:

1. The binary operation is commutative.
2. The semi-group is finitely generated.
3. S can be embedded into a lattice.

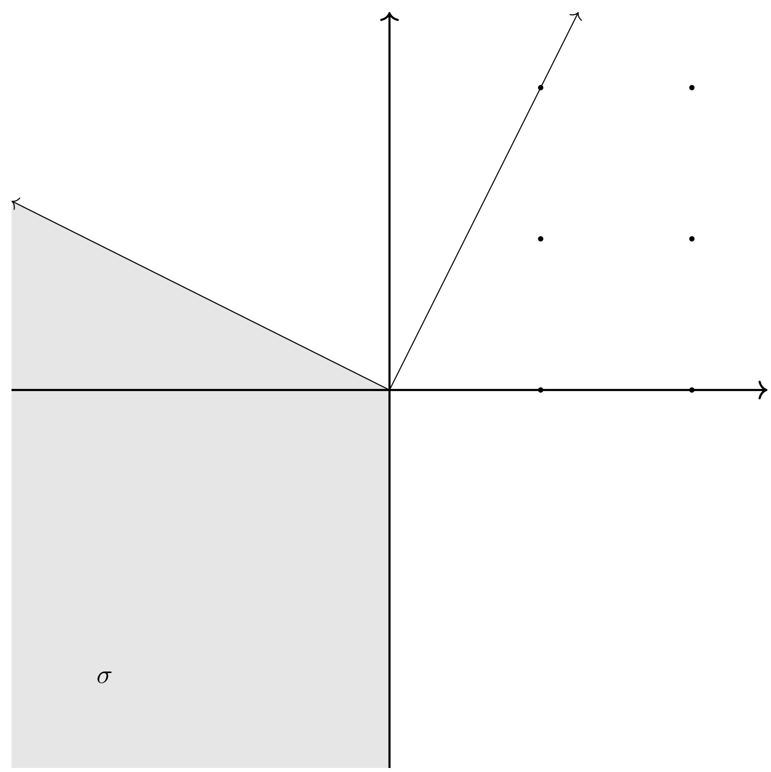
Example 6. Let $\sigma := \mathrm{vcone}\begin{pmatrix} -1 & \\ -1 & -1 \end{pmatrix}$.



One can read from the picture that

$$S_\sigma = \left\langle \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle.$$

Example 7. Let $\sigma := \operatorname{vcone}\begin{pmatrix} -2 & \\ 1 & -1 \end{pmatrix}$.



One can read from the picture that

$$S_\sigma = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle.$$

Definition. A (commutative, finitely generated) semigroup is a set S with an associative and commutative binary operation and an identity element, that can be embedded into a lattice.

Proposition 6 (Gordan's Lemma)**.** *If σ is a cone in N, then S_σ is a finitely generated semi-group.*

Proof. Let $\sigma^\vee = \operatorname{vcone}(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s)$, and let $K := \{x_1\boldsymbol{v}_1 + \cdots + x_s\boldsymbol{v}_s \mid 0 \le x_i < 1\}$. Then $\sigma^\vee \cap M$ is generated by $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s\} \cap K$. $\square$

Definition. Let σ be a pointed cone. Consider $0 \neq m \in \sigma^\vee \cap M = S_\sigma$, it is called *irreducible* if for any decomposition $m = k + l$ in S_σ, either $k = 0$ or $l = 0$.

Proposition 7. *Let σ be a pointed polyhedral cone in $\mathbb{R}^n$, and let*

$$H := \{m \in S_\sigma \mid m \textit{ is irreducible}\},$$

then

1. $|H| < \infty$.
2. *H generates S_σ.*
3. *Every generating set contains H.*

Here the set is called the Hilbert basis.

Definition. A polytope P is said to be *simplicial* if all its facets are simplices.

Definition. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is said to be *normal* if

$$(kP) \cap M + (lP) \cap M = ((k+l)P) \cap M$$

for all $k, l \in \mathbb{N}$.

Theorem 8. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope of dimension $n \geq 2$, then kP is normal for all $k \geq n-1$.*

Definition. A sub-semi-group $S \subseteq M$ is said to be *saturated* if whenever $m \in M$ and $pm \in M$ for some $p \in \mathbb{N}_+$, $m \in S$.

Definition. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is said to be *very ample* if for every vertex $m \in P$, the semi-group $S_{P,m} := \mathbb{N}\langle P \cap M - m\rangle$ is saturated in M.

Proposition 9. *A normal lattice polytope is very ample.*

Definition. A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones such that:

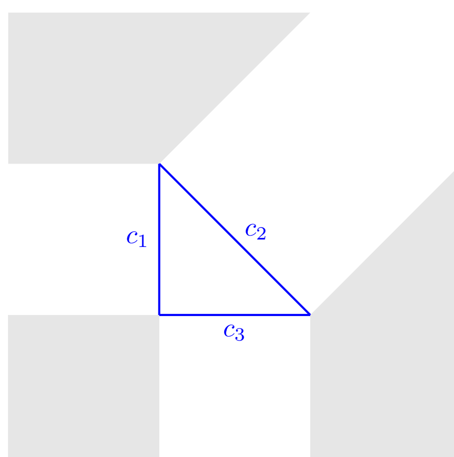
1. Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
2. For all $\sigma \in \Sigma$, each face of σ is also in Σ.
3. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Definition. Given a full-dimensional lattice polytope P in $\mathbb{R}^n$, for each face F of the polytope, define

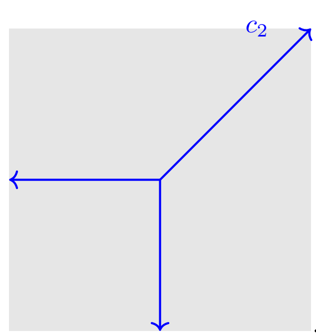
$$\sigma_F := \{\boldsymbol{u} \in N_{\mathbb{R}} \mid \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{u}\rangle \leq 0, \forall \boldsymbol{x} \in F, \boldsymbol{y} \in P\}.$$

Let $\Sigma_P := \{\sigma_F \mid F \subseteq P \text{ is a face}\}$, then Σ_P is a fan, called the normal fan associated with the polytope P.

Example 8. Given the polytope



we shall have the normal fan



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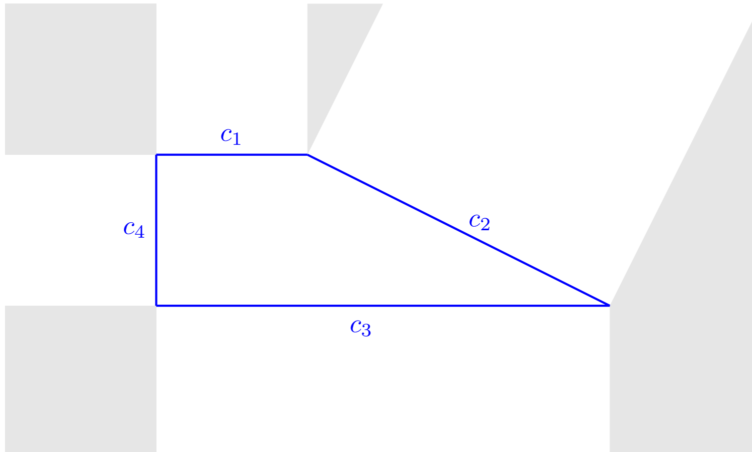
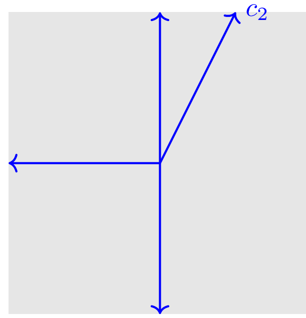
Example 9. Given the polytope



we shall have the normal fan



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4 Toward Toric Varieties

Let σ be a rational polyhedral cone in $N_{\mathbb{R}} \cong \mathbb{R}^n$. Here comes our main construction:

Definition. Let $A_\sigma := \mathbb{C}[S_\sigma]$ be a $\mathbb{C}$-algebra, such that

1. $\{t^m \mid m \in S_\sigma\}$ forms a $\mathbb{C}$-basis for A_σ.
2. $t^{m_1}t^{m_2} = t^{m_1+m_2}$.

We then call $U_\sigma :=$ Spec A_σ the affine toric variety associated to σ. Notice that this construction can be generalised to a semigroup, where $\mathbb{C}[S]$ is called the semigroup algebra for the semigroup S.

If $S_\sigma = \langle m_1, \cdots, m_r \rangle$, then A_σ is generated (as a ring) by $t^{m_1}, \cdots, t^{m_r}$. In particular A_σ is Noetherian.

Example 10. Let $\sigma :=$ vcone $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. Then $S_\sigma = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$ and thus $\mathbb{C}[S_\sigma] = \mathbb{C}[s,t]$, $U_\sigma = \mathbb{C}^2$. Similarly, if $\sigma :=$ vcone$(-I_n)$, then $U_\sigma = \mathbb{C}^n$.

Example 11. Let $\sigma :=$ vcone $\begin{pmatrix} -1 & \\ -1 & -1 \end{pmatrix}$. Then $S_\sigma = \left\langle \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$ and thus $\mathbb{C}[S_\sigma] = \mathbb{C}[s^{-1}t^{-1}, t] = \mathbb{C}[x,y]$, $U_\sigma = \mathbb{C}^2$.

Example 12. Let $\sigma :=$ vcone $\begin{pmatrix} -2 & \\ 1 & -1 \end{pmatrix}$. From previous computation we know that $A_\sigma = \mathbb{C}[s, st, st^2] = \mathbb{C}[x,y,z]/(y^2 - xz)$.

Question: the previous examples have 2 generators and 3 generators respectively. What are the differences?

Example 13. Let $\sigma :=$ vcone(0), then $S_\sigma = \mathbb{Z}^2$ and $A_\sigma = \mathbb{C}[s, s^{-1}, t, t^{-1}]$, hence U_σ is the 2-torus.

The toric variety defined in Theorem 2 is a Zariski closed subset in an affine space, so it is an affine variety. Therefore we would like to find its defining ideal. As in the proof, we have an induced $\hat{\Phi}_A : \mathbb{Z}^s \to M$. Let K be its kernel then there is a short exact sequence

$$0 \to K \to \mathbb{Z}^s \to M.$$

An element $k = (k_1, \cdots, k_s) \in K$ satisfies $\sum_{i=1}^s k_i m_i = 0$, i.e. a linear relation among the m_i. Let $k_+ := \sum_{k_i>0} k_i e_i$ and let $k_- := -\sum_{k_j<0} k_j e_j$, then $k = k_+ - k_-$ and $k_+, k_- \in \mathbb{N}^s$. The binomial

$$x^{k_+} - x^{k_-} = \prod_{k_i>0} x_i^{k_i} - \prod_{k_j<0} x_j^{-k_j}$$

vanished on the image of Φ_A, because

$$\prod_{k_i>0} (\chi^{m_i})^{k_i} - \prod_{k_j<0} (\chi^{m_j})^{-k_j} = \prod_{k_j<0} (\chi^{m_j})^{-k_j} \left(\prod_{i=1}^{s} x^{k_i m_i} - 1 \right) = 0.$$

So the ideal contains the set

$$\langle x^{k_+} - x^{k_-} \mid k \in K \rangle = \langle x^m - x^n \mid m, n \in \mathbb{N}^s \text{ and } m - n \in K \rangle.$$

In fact, we have

Theorem 10. *The ideal of the toric variety X_A for $A = \{m_1, \cdots, m_s\} \subseteq M$ is*

$$\langle x^{k_+} - x^{k_-} \mid k \in K \rangle = \langle x^m - x^n \mid m, n \in \mathbb{N}^s \textit{ and } m - n \in K \rangle.$$

Definition. Let $K \subseteq \mathbb{Z}^s$ be a sublattice.

1. The ideal $\langle x^m - x^n \mid m, n \in \mathbb{N}^s$ and $m - n \in K \rangle$ is called a lattice ideal.
2. A prime lattice ideal is called a toric ideal.

Hence by Theorem 10, the defining ideal for the toric variety X_A is a toric ideal.

Proposition 11. *An ideal $I \subseteq \mathbb{C}[x_1, \cdots, x_s]$ is toric if and only if it is prime and generated by binomials.*

We then consider a semigroup S in M, then we have a semigroup algebra $\mathbb{C}[S]$. It is an integral domain and finitely generated as a $\mathbb{C}$-algebra.

Proposition 12. *The variety $\operatorname{Spec} \mathbb{C}[S]$ is a toric variety. If the semigroup $S = \mathbb{N}A$ is generated by the set $A = \{m_1, \cdots, m_s\}$, then*

$$\operatorname{Spec} \mathbb{C}[S] \cong X_A.$$

Theorem 13. *This construction gives us an affine toric variety and all affine toric varieties (probably not normal) come from this way (from a semigroup). More precisely, if V is an affine variety, then the following are equivalent:*

1. *V is an affine toric variety.*
2. *$V \cong X_A$ for a finite set A in a lattice.*
3. *V is an affine variety defined by a toric ideal.*
4. *$V = \operatorname{Spec} \mathbb{C}[S]$ for an affine semigroup S.*

Proposition 14. *An affine toric variety has a fixed point of the torus action if and only if*

Definition. Let σ be a pointed cone in $N = \mathbb{Z}^n$. We say

1. σ is *simplicial* if the number of extremal rays $= \dim \sigma$.
2. σ is *smooth* if σ is generated by a part of a $\mathbb{Z}$-basis of N.

Recall that a $\mathbb{C}$ point of a variety X is a morphism $\mathbb{C} \to X$. When $X = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ is an affine toric variety, it is given by the associated ring map

$$\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}.$$

This has to come from a semi-group homomorphism:

Proposition 15. *Let $V = \operatorname{Spec} \mathbb{C}[S]$ be the affine toric variety of the affine semi-group S. Then there is a correspondence between the following:*

1. *$\mathbb{C}$-points in V.*
2. *Maximal ideals $\mathfrak{m}$ in $\mathbb{C}[S]$.*
3. *Semi-group homomorphisms $S \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semi-group under multiplication.*

Theorem 16. *Let σ be a pointed cone in N and $\dim \operatorname{Span} \sigma = k$, then the following are equivalent:*

1. *U_σ is smooth.*
2. *$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$.*
3. *σ is smooth.*

Theorem 17. *For an affine toric variety X, the following are equivalent:*

1. *X is normal.*
2. *$X := \operatorname{Spec} \mathbb{C}[S]$ where S is a saturated affine semi-group.*
3. *$X \cong U_\sigma$ for some strongly convex rational polyhedral cone.*

Proposition 18. *For the toric variety $X_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$,*

$$\dim T_{X_\sigma, P_\sigma} = |H|$$

where H is the Hilbert basis of $\sigma^\vee \cap M$.

Proposition 19. *Let $\tau = \sigma \cap u^\perp$ be a face of σ where $u \in \sigma^\vee$. Then the semi-group algebra $\mathbb{C}[S_\tau]$ is the localization of $\mathbb{C}[S_\sigma]$ at the point $t^m \in \mathbb{C}[S_\sigma]$.*

Actually Proposition 19 gives us the information which is called the gluing data. Suppose τ is a common face of σ_1, σ_2, then we have

$$U_{\sigma_1} \hookleftarrow U_\tau \hookrightarrow U_{\sigma_2},$$

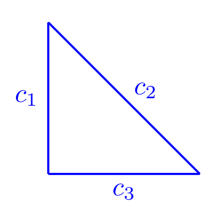
and we can glue U_{σ_1} and U_{σ_2} along U_τ. But where can we find the structures including common faces? The definition of a fan and here is also where polytopes can be related.

Proposition 20. *If $\sigma_1, \sigma_2 \in \Sigma$ and $\tau = \sigma_1 \cap \sigma_2$, then*

$$S_\tau = S_{\sigma_1} + S_{\sigma_2}.$$

Proof. The inclusion $S_{\sigma_1} + S_{\sigma_2} \subseteq S_\tau$ follows directly from the general fact that $\sigma_1^\vee + \sigma_2^\vee = (\sigma_1 \cap \sigma_2)^\vee = \tau^\vee$. ∎

Example 14. Consider the normal fan of the polytope $P =$



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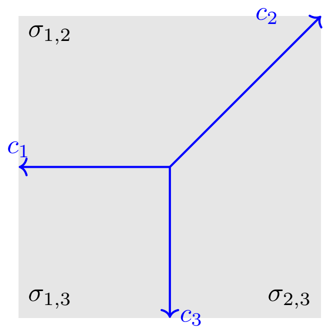
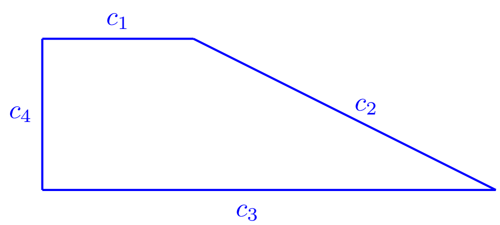
then by previous computation, the associated fan

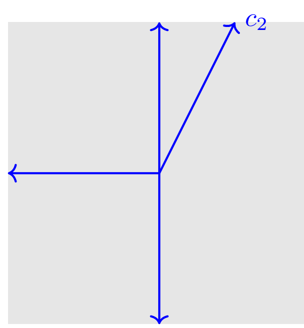


has affine pieces $U_{1,2} = \text{Spec } \mathbb{C}[s^{-1}, st] = \mathbb{C}^2, U_{1,3} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}] = \mathbb{C}^2, U_{1,3} = \text{Spec } \mathbb{C}[st, t^{-1}] = \mathbb{C}^2$, and gluing affine open subsets $U_1 = \mathbb{C}[s^{-1}, st]_{s^{-1}} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}]_{s^{-1}}$, $U_2 = \text{Spec } \mathbb{C}[s^{-1}, st]_{st} = \mathbb{C}[st, t^{-1}]_{st}$, and $U_3 = \mathbb{C}[s^{-1}, t^{-1}]_{t^{-1}} = \mathbb{C}[st, t^{-1}]_{t^{-1}}$. This gives the variety $\mathbb{CP}^2$.

Example 15. Consider the normal fan of the polytope $P =$



then by previous computation, the associated fan



has affine pieces $U_{1,2} = \text{Spec } \mathbb{C}[st^2, t] = \mathbb{C}^2, U_{1,4} = \text{Spec } \mathbb{C}[s^{-1}, t] = \mathbb{C}^2, U_{3,4} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}] = \mathbb{C}^2, U_{2,3} = \text{Spec } \mathbb{C}[st^2, t^{-1}] = \mathbb{C}^2$, and gluing affine open subsets $U_1 = \mathbb{C}[t, st^2]_t = \text{Spec } \mathbb{C}[s^{-1}, t]_t$, $U_2 = \text{Spec } \mathbb{C}[st^2, t]_{st^2} = \text{Spec } \mathbb{C}[st^2, t^{-1}]_{st^2}, U_3 = \text{Spec } \mathbb{C}[st^2, t^{-1}]_{t^{-1}} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}]_{t^{-1}}$, and $U_4 = \mathbb{C}[s^{-1}, t^{-1}]_{s^{-1}} = \mathbb{C}[t, s^{-1}]_{s^{-1}}$. This gives the variety at the very beginning. The surface is called the Hirzebruch surface $\mathbb{F}_2$.

In conclusion, we have the following

Theorem 21. *Given a fan Σ, there is a toric variety X_Σ associated with the fan, which is separated and normal.*

Proof. We omit the verification of compatibility. The variety is normal because we know there is an affine open cover where each affine open is normal.

To see X_Σ is separated, it suffices to show that for each pair of cones σ_1, σ_2 in Σ, the image of the diagonal map

$$\Delta : U_\tau \to U_{\sigma_1} \times U_{\sigma_2}, \tau = \sigma_1 \cap \sigma_2$$

is Zariski closed. But Δ comes from the $\mathbb{C}$-algebra homomorphism

$$\Delta^* : \mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}] \to \mathbb{C}[S_\tau]$$

given by $\chi^m \otimes \chi^n \mapsto \chi^{m+n}$. By Proposition 20, Δ^* is surjective, so that

$$\mathbb{C}[S_\tau] \cong (\mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}])/I,$$

hence the image of Δ is Zariski closed. $\square$

Theorem 22. *Given a fan Σ, denote by X_Σ the toric variety associated with the fan. Then*

1. *All normal Toric varieties arise from a fan.*
2. *X_Σ is smooth if and only if Σ is smooth, i.e. all the cones are smooth.*
3. *X_Σ is simplicial (i.e. X_Σ is an orbifold, having only finitely many quotient singularities) if and only if Σ is simplicial. When the fan Σ comes from a polytope, this is equivalent to that the polytope is simplicial.*
4. *X_Σ is complete if and only if the support $|\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma$ is all of $N_\mathbb{R}$.*

5 Where is the Action?

We assume in this section $\dim\sigma = \operatorname{rank} N$ is the maximal. Notice that we always have $\sigma \subseteq N$, or $\sigma^\vee \subseteq M$, which leads to a ring homomorphism

$$\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[M]$$

and thus

$$\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[M] \otimes_\mathbb{C} \mathbb{C}[\sigma^\vee \cap M].$$

Recall that a G action is a morphism $\sigma : G \times X \to X$ satisfying

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{(\mu,\mathrm{id}_X)} & G\times X \\ {\scriptstyle(\mathrm{id}_G,\sigma)}\downarrow & & \downarrow{\scriptstyle\sigma} \\ G\times X & \xrightarrow{\quad\sigma\quad} & X, \end{array}$$

where μ is the multiplication.

Similarly, for a toric variety given by a fan, every affine piece and every intersection contains the torus given by M, so we have the torus acting on the variety.

6 Orbit-Cone Correspondence

We use a concrete example to illustrate this. Consider in Example 14, the toric variety is $X_\Sigma = \mathbb{CP}^2$, where the torus is $T^2 \subseteq \mathbb{CP}^2$ with homogeneous coordinates $(1, s, t)$ s.t. $s, t \neq 0$ (there are also other embeddings). For each $u = (a, b) \in M \cong \mathbb{Z}^2$, we have the corresponding curve (actually an 1-parameter subgroup λ^u composed with the embedding $T_M \hookrightarrow \mathbb{CP}^2$):

$$\lambda^u(t) = (1, t^{-a}, t^{-b}).$$

Consider the limit point as $t \to 0$, for instance, if $a < 0, b < 0$, then $\lim_{t\to0}(1, t^{-a}, t^{-b}) = (1, 0, 0)$. If $a = b > 0$, then $\lim_{t\to0}(1, t^{-a}, t^{-b}) = (0, 1, 1)$. Some further computations tell us that we can just recover the structure of the fan by taking limits!

Back to T_M acting on $\mathbb{CP}^2$, there are exactly seven orbits:

$$\begin{aligned} O_1 = & \\ & O_2 \\ & O_3 \\ & O_4 \\ & O_5 \\ & O_6 \\ & O_7. \end{aligned}$$

This list shows that each orbit contains a unique limit point. Hence we obtain a correspondence between cons and orbits. In general, we have:

Theorem 23. *Let X_Σ be the toric variety of the fan $\Sigma \subseteq N$. Then:*

1. *There is a bijective correspondence*

$$\begin{aligned}\{\textit{cones } \sigma \textit{ in } \Sigma\} &\longleftrightarrow \{T_N\textit{-orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma) \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\vee \cap M, \mathbb{C}^*).\end{aligned}$$

2. *Let $n = \dim N_{\mathbb{R}}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.*

3. *The affine open subset U_σ is the union of orbits*

$$U_\sigma = \bigcap_{\tau \leq \sigma} O(\tau).$$

4. *$\tau \leq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\sigma)}$, and*

$$\overline{O(\tau)} = \bigcap_{\tau \leq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski Topologies.

7 Slogan: In Toric World, Geometries are Combinatorics

Theorem 24. *A normal toric variety is Cohen-Macaulay.*

When the polytope is very ample, we not only get a projective toric variety, but also the morphism how it is embedded into a projective space. This information is called a very ample line bundle.

Theorem 25 (Ehrhart)**.** *We have seen that the Ehrhart series for a full dimensional polytope $P \subseteq \mathbb{R}^n$*

$$\begin{aligned}\mathrm{Ehr}(P, -) : \mathbb{N} &\to \mathbb{N} \\ t &\mapsto \#(tP \cap \mathbb{Z}^n)\end{aligned}$$

is a polynomial with a reciprocity

$$\mathrm{Ehr}(P, -t) = (-1)^d \mathrm{Ehr}(P, t)$$

where d is the degree of the polynomial.

This theorem can be proved using toric varieties. Let V be the (projective) toric variety associated with the polytope P, then (some enlarged) the polytope gives an ample line bundle L over V. Then the Ehrhart series coincides with the Hilbert series of this line bundle almost by definition. Since the Hilbert series is a polynomial, we are done.

Theorem 26 (Stanley, 86', [2])**.** *Let P be a n-dimensional simplicial polytope, and let the f-vector $f = (f_0, \cdots, f_{n-1})$ be a sequence of numbers where f_j is the number of j-faces of P. Let $f_{-1} = 1$. Define*

$$h_i := \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}$$

then we have the so called h-vector. The Dehn-Sommerville equations say that

$$h_i = h_{n-i}, \forall 1 \leq i \leq n,$$

which hold for any simplicial convex polytope. A sequence of integers $(k_0, \cdots, k_n)$ is said to be an M-vector (after F.S.Macauley) if

$$k_0 = 1 \textit{ and } k_{i+1} \leq k_i^{\langle i \rangle} \textit{ for all } 1 \leq i \leq n-1,$$

where $k_i^{\langle i \rangle}$ is defined to be

$$k_i^{\langle i \rangle} := \binom{n_i + 1}{i+1} + \cdots \binom{n_j + 1}{j+1},$$

where $n_i \geq n_{i-1} \geq n_j \geq j \geq 1$ are those (unique) numbers such that

$$k_i = \binom{n_i}{i} + \cdots \binom{n_j}{j}.$$

A sequence of integers $(h_0, \cdots, h_n)$ is the h-vector of a simplicial convex n-polytope if and only if $h_0 = 1$, $h_i = h_{n-i}$ and the sequence $(h_0, h_1 - h_0, \cdots, h_{[n/2]} - h_{[n/2]-1})$ is an M-vector.

This was originally conjectured by McMullen [3], and proved by Stanley using some great toric tools (Hard Lefschetz).

8 The Proofs

Proof of Theorem 2. By Proposition 1, the image $T = \Phi_A((\mathbb{C}^*)^n)$ is a torus, which is also closed in $(\mathbb{C}^*)^s$. Thus $T = X_A \cap (\mathbb{C}^*)^s$ since X_A is the Zariski closure. Furthermore, it is a torus so it is irreducible, so is X_A.

We next consider the action. Since $T \subseteq (\mathbb{C}^*)^s$, an element $t \in T$ acts on $(\mathbb{C}^*)^s$ and takes varieties to varieties. Notice that

$$T = t \cdot T \subseteq X_A,$$

which means $t \cdot X_A$ is also a variety containing T. Thus $X_A \subseteq t \cdot X_A$ because X_A is the Zariski closure. Replacing t by t^{-1} leads to $t \cdot X_A \subseteq X_A$, so $X_A = t \cdot X_A$. Therefore X_A is a toric variety.

It remains to compute the character lattice. We have a commutative diagram of tori

$$\begin{tikzcd} T_M \arrow[r, "\Phi_A"] \arrow[dr] & (\mathbb{C}^*)^s \\ & T, \arrow[u, hook] \end{tikzcd}$$

inducing a diagram of character lattice

$$\begin{tikzcd} M & \mathbb{Z}^s \arrow[l, "\hat{\Phi}_A"'] \arrow[d] \\ & L \arrow[ul, hook] \end{tikzcd}$$

where L denotes the character lattice. Since $\hat{\Phi}_A$ takes the standard basis $e_1, \cdots, e_s$ to $m_1, \cdots, m_s$, thus the image of $\hat{\Phi}_A$ is $\mathbb{Z}A$. Hence $L \cong \mathbb{Z}A$. $\square$

Proof of Theorem 10. Let I denote the ideal, then $I \subseteq I(X_A)$. Pick a monoidal order $\geq$ on $\mathbb{C}[x_1 \cdots, x_s]$. The map $\Phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^s$ is given by the Laurent monomial t^{m_i} in variables $t_1, \cdots, t_n$. If $I \neq I(X_A)$, then we can pick $f \in I(X_A) - I$ with minimal leading monomial $x^\alpha = \prod_{i=1}^s x_i^{a_i}$. Rescaling if necessary, let x^α be the leading term (i.e. with coefficient 1).

However $f(t^{m_1}, \cdots, t^{m_s}) : (\mathbb{C}^*)^n \to \mathbb{C}^s \to \mathbb{C}$ is identically 0 as a (series hence a) polynomial in $t_1, \cdots, t_n$, there must be cancellation invovling the term coming from x^α. In other words, f must contain a monomial $x^\beta = \prod_{i=1}^s x_i^{b_i} < x^\alpha$ such that

$$\prod_{i=1}^s (t^{m_i})^{a_i} = \prod_{i=1}^s (t^{m_i})^{b_i}.$$

This implies $\sum_{i=1}^s a_i m_i = \sum_{i=1}^s b_i m_i$, so that $\alpha - \beta \in K$. Then $x^\alpha - x^\beta \in I$, and it follows that $f - x^\alpha + x^\beta$ also lies in $I(X_A) - I$, with a strictly smaller leading term. Hence we get a contradiction. $\square$

Proof of Proposition 11. One direction is easy. Suppose I is generated by binomials $x^{\alpha_i} - x^{\beta_i}$ and prime, then $V(I) \subseteq \mathbb{C}^s$ is irreducible. Observe that $V(I) \cap (\mathbb{C}^*)^s \neq \emptyset$ (since $(1, \cdots, 1)$ is in the intersection), implying that $V(I) \cap (\mathbb{C}^*)^s$ is a subvariety hence a torus.

By projecting on the i-th coordinate, there is a character $T \hookrightarrow (\mathbb{C}^*)^s \to \mathbb{C}^*$, which is χ^{m_i} by our usual convention, where m_i is given by α_i and β_i. The construction gives that T is the image of

$$\Phi_A(t) = (\chi^{m_1}(t), \cdots, \chi^{m_s}(t)),$$

hence $V(I) = X_A$ by the irreduciblity. Therefore $I = I(V(I))$ is toric by Nullstellensatz and Theorem 10. $\square$

Proof of Proposition 12. Using $A = \{m_1, \cdots, m_s\}$, we get a $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[x_1 \cdots, x_s] \to \mathbb{C}[M]$$

by $x_i \mapsto \chi^{m_i}$. This corresponds to the morphism

$$\Phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^s,$$

i.e. it is $(\Phi_A)^*$. The kernel of this map is $I(X_A)$ and the image is

$$\mathbb{C}[\chi^{m_1}, \cdots, \chi^{m_s}] = \mathbb{C}[S].$$

Thus the coordinate ring of X_A is

$$\mathbb{C}[x_1 \cdots, x_s]/I(X_A) = \mathbb{C}[S],$$

which means these two affine varieties are isomorphic. □

The torus multiplication rises to an action $(\mathbb{C}^*)^n \circlearrowleft \mathbb{C}[M]$ defined by

$$f \mapsto t \cdot f : (p \mapsto f(t^{-1} \cdot p)).$$

Lemma 5. *Let V be a subspace stable under the action of $(\mathbb{C}^*)^n$. Then*

$$V \cong \bigoplus_{\chi^m \in V} \mathbb{C} \cdot \chi^m.$$

Proof. Let $W := \bigoplus_{\chi^m \in V} \mathbb{C} \cdot \chi^m$, then apparently $W \subseteq V$. For the opposite inclusion, pick $0 \neq f \in V$, we write

$$f = \sum_{m \in B} c_m \chi^m,$$

where $B \subseteq M$ is finite and $c_m \neq 0$ for all m. Then $f \in D \cap W$ where

$$D := \operatorname{span}(\chi^m \mid m \in B) \subseteq \mathbb{C}[M].$$

Since $t \cdot \chi^m = \chi^m(t^{-1})\chi^m$, D and hence $D \cap V$ are stable under the action of T_N. But $D \cap V$ is finite-dimensional, so one can show that $D \cap V$ is spanned simultaneous eigenvectors of T_N. This is taking place in $\mathbb{C}[M]$, where simultaneous eigenvectors are characters. Thus $D \cap V$ is spanned by characters. The above expression for $f \in D \cap V$ implies $\chi^m \in V$ for $m \in B$. Hence $f \in W$. □

Now we turn to the proof of equivalence of constructions:

Proof of Theorem 13. The implications $2 \Leftrightarrow 3 \Leftrightarrow 4 \Rightarrow 1$ follows from Theorem 2, Theorem 10, and Proposition 12. For $1 \Rightarrow 4$, let X be a toric variety containing the torus $(\mathbb{C}^*)^n$ with character lattice M. Since the coordinate ring of $(\mathbb{C}^*)^n$ is a semi-group algebra $\mathbb{C}[M]$, the inclusion $(\mathbb{C}^*)^n \hookrightarrow X$ induces a ring homomorphism

$$\mathbb{C}[X] \to \mathbb{C}[M].$$

Since $(\mathbb{C}^*)^n$ in X is Zariski dense, this homomorphism is injective, so we can view $\mathbb{C}[X]$ as a subalgebra of $\mathbb{C}[M]$. Since the action of $(\mathbb{C}^*)^n$ on X is given by a morphism

$$(\mathbb{C}^*)^n \times X \to X,$$

then for $t \in (\mathbb{C}^*)^n$ and $f \in \mathbb{C}[X]$, $f : p \mapsto f(t^{-1} \cdot p)$ is a morphism on X. It follows that $\mathbb{C}[X]$ is stable under the action of $(\mathbb{C}^*)^n$. Thus by the previous lemma,

$$\mathbb{C}[X] \cong \bigoplus_{\chi^m \in \mathbb{C}[X]} \mathbb{C} \cdot \chi^m,$$

which means $\mathbb{C}[X]$ is the semi-group algebra for the semi-group $S = \{\chi^m \in \mathbb{C}[X]\}$. □

Proof of Theorem 17. $1 \Rightarrow 2$: If X is normal, then $\mathbb{C}[X] = \mathbb{C}[S]$ is integrally closed in the field $\mathbb{C}(X)$. Suppose $km \in S$ for some $k \in \mathbb{N}^*$ and $m \in M$, then χ^m is a polynomial function on $(\mathbb{C}^*)^n$ and hence a rational function on X since $(\mathbb{C}^*)^n \subseteq X$ is Zariski open dense. We also have $\chi^{km} \in \mathbb{C}[S]$. It follows that χ^m is a root of the monic polynomial

$$X^k - \chi^{km} \in \mathbb{C}[S][X].$$

Therefore $\chi^m \in \mathbb{C}[S]$, i.e. $m \in S$.

$2 \Rightarrow 3$: Let A be a finite generating set of the semi-group S, then S lies in the polyhedral cone $\operatorname{cone}(A) \subseteq M_{\mathbb{R}}$. $\operatorname{rank} \mathbb{Z}A = n$ implies $\dim \operatorname{cone}(A) = n$. It follows $\sigma = \operatorname{cone}(A)^\vee$ is a strongly convex rational cone such that $S \subseteq \sigma^\vee \cap M$. By some combinatorics, the equality holds.

$3 \Rightarrow 1$: We need to show that $\mathbb{C}[\sigma^\vee \cap M]$ is normal if σ is a strongly convex rational polyhedral cone. Let $\rho_1, \cdots, \rho_r$ be the rays of σ. Since it is generated by the rays, we have

$$\sigma^\vee = \bigcap_{i=1}^{r} \rho_i^\vee,$$

and intersecting with M implies $S_\sigma = \bigcap_{i=1}^r S_{\rho_i}$, also gives

$$\mathbb{C}[S_\sigma] = \bigcap_{i=1}^{r} \mathbb{C}[S_{\rho_i}].$$

Hence it suffices to prove each $\mathbb{C}[S_{\rho_i}]$ is normal when ρ is a rational ray in $N_{\mathbb{R}}$. Let u be its generator, and assume it is primitive, i.e. $\frac{1}{k}u \notin N$. Hence we can extend this to a basis $e_1, \cdots, e_n$ of N s.t. $u = e_1$. Thus (under an isomorphism induced by basis change)

$$\mathbb{C}[S_\rho] \cong \mathbb{C}[x_1, x_2^{\pm}, \cdots, x_n^{\pm}].$$

$\mathbb{C}[x_1, \cdots, x_n]$ is a UFD, hence it is normal, and so is its localisation $\mathbb{C}[x_1, x_2^{\pm}, \cdots, x_n^{\pm}] = \mathbb{C}[x_1, \cdots, x_n]_{x_2 \cdots x_n}$. □

Theorem 27 (Sumihiro)**.** *Let the torus T_N act on a normal separated variety X. Then every point $p \in X$ has a T_N invariant affine open neighborhood.*

Proof sketch of Theorem 22. By Theorem 27, we have an affine open cover U_{σ_i} for some cone σ_i. Then using Orbit-Cone correspondence, we are able to show $U_i \cap U_j$ is the affine toric variety associated to the cone $\tau = \sigma_i \cap \sigma_j$. Then (use Orbit-Cone correspondence instead of just combinatorics) one can prove that τ is a face of both σ_i and σ_j. □

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