

Some Open Problems I am Interested in

Guanyu Li

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I maintain this document only to keep a record of the problems that I was/am interested in. Once a problem was wrote, I would not mark it again. Hence there is possibility that the problem is not a problem anymore. Also, you can see a lot of chaos in this document, not in order, so unrelated, classified into numerous fields. However, I hope this document can make me feel less aimless, and can probably help others find interesting problems.

Problem 1 (Kervaire Invariant Problem in 126, Geometrical Topology). In 1960, Kervaire proved that there is a manifold homotopic to a sphere S^2 without any smooth structure [4]. What he did was actually constructing a "Kervaire invariant" that lived in $\pi_2^s(S^0)$. In 1969, Browder proved that the Kervaire invariant could only live in $\pi_{2^n-2}^s(S^0)$ [1] and soon Barratt, Jones, Mahowald proved for $n = 2, 3, 4, 5, 6$ the fundamental group was not trivial and the element existed [2]. Since then 126 has become an open problem for the existence. In 2016, Hill, Hopkins and Ravenel proved $\pi_{256k-2}^s(S^0) = 0$ so that the invariant does not exist for $n \geq 8$. Thus the only left one is $n = 7$ [3].

Problem 2 (Auto-equivalence of Derived Categories, Derived Algebraic Geometry). Bridgeland, sm 3-folds

Problem 3 (Class Number Problem, Number Theory). When we were solving number theoretical equations like

$$x^2 + dy^2 = n$$

for some integer n , it is helpful for consider the primes in $\mathbb{Z}[\sqrt{-d}]$. Since generally $\mathbb{Z}[\sqrt{-d}]$ is not a U.F.D., it is better to consider its integral closure \mathcal{O}_K where $K = \mathbb{Q}(\sqrt{-d})$. This ring \mathcal{O}_K is a Dedekind domain, Gauss, in his famous book *Disquisitiones Arithmeticae* of 1801 (Section V, Articles 303 and 304), posed these questions

Problem 4 (The Section Conjecture, Algebraic Geometry). When we were dealing with a scheme X , the usual construction of the fundamental group fails since it is difficult to have a path in X . However by the relations between the covering spaces and their deck transformation groups, we are able to have a notion of étale fundamental group $\pi_1^{\text{ét}}(X)$.

Problem 5 (Moduli of Mixed Hodge Structure, Algebraic Geometry). Kapranov

Problem 6 (Tannakian Reconstruction, Algebraic Geometry). If X is a G -variety where G is a reductive group, and $x \in X$ is a point with a dense orbit. We would get (by beyond GIT) a complex $DF(X/G, x)_\bullet$ (degeneration fan). If X is toric, then DF recovers the fan of X . Then the question is to what extent does DF determine X ?

Problem 7 (Quasi-smooth Hilbert Scheme, DAG). We have the Hilbert scheme in the usual algebraic geometry. However, in the derived world, the analogy is to consider the moduli of (derived) l.c.i. families of subschemes.

References

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